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SHARP SPECTRAL MULTIPLIERS FOR OPERATORS SATISFYING GENERALIZED GAUSSIAN ESTIMATES

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ABSTRACT. Let L be a non-negative self adjoint operator acting on $L^2(X)$ where X is a space of homogeneous type. Assume that L generates a holomorphic semigroup e^{-tL} whose kernels $p_t(x,y)$ satisfy generalized m-th order Gaussian estimates. In this article, we study singular and dyadically supported spectral multipliers for abstract self-adjoint operators. We show that in this setting sharp spectral multiplier results follow from Plancherel or Stein-Tomas type estimates. These results are applicable to spectral multipliers for large classes of operators including m-th order elliptic differential operators with constant coefficients, biharmonic operators with rough potentials and Laplace type operators acting on fractals.

CONTENTS

1. INTRODUCTION

This paper is devoted to the theory of spectral multipliers of self-adjoint differential type operators. This is a classical area of harmonic analysis, which has attracted a lot of attention during in the last fifty years or so. The literature devoted to the subject is so broad that it is impossible to provide complete and comprehensive bibliography. Therefore we quote only a few papers, which are directly related to our study and refer readers to [4, 11, 12, 13, 16, 17, 21, 22, 24, 25, 29, 30, 31, 32, 35, 41, 42, 45, 46, 49, 50] and the references within for further relevant literature.

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We consider a measure space X and a non-negative self-adjoint operator L acting on $L^2(X)$. Such an operator admits a spectral resolution $E_L(\lambda)$ and for any bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$, one can define the operator $F(L)$

$$F(L) = \int_0^\infty F(\lambda)\, \mathrm{d}E_L(\lambda). \tag{1.1}$$

By spectral theory the operator $F(L)$ is bounded on $L^2(X)$ and its norm is equal to L-essential supremum norm of F. Spectral multiplier theorems investigate under what conditions on function F the operator $F(L)$ can be extended to a bounded operator acting on Lebesgue spaces $L^p(X)$ for some range of p. Usually one looks for condition formulated in terms of differentiability of function F. Spectral multiplier theorems are closely related to the problem of Bochner-Riesz sumability. There are some subtle differences between two problems but the essential core of Bochner-Riesz sumability problem and the spectral multiplier theorems is identical.

We would like to mention three different aspects of spectral multipliers theory and Bochner-Riesz analysis.

• Dyadically supported spectral multipliers. Here one assumes that a function $F \in C_c(a,b)$ for some $0 < a < b$ is compactly supported and one tries to find necessary conditions to ensure that

$$\sup_{t>0} \|F(tL)\|_{p\to p} \le C < \infty$$

for some $p \in [1,\infty]$ or range of such p, where $F(tL)$ is defined by the spectral resolution. Usually the condition of F is expressed in terms of Sobolev spaces $W^s_q(\mathbb{R})$ and constant C is proportional to the corresponding Sobolev norm of function F. Compact support assumption could be misleading here as it can be essentially weakened but it is convenient because the dyadic decomposition trick is often used in the theory of spectral multipliers. Usually the proof of sharp results of this type requires Plancherel or restriction type estimates, which we discuss below, see Section 4.

• Singular integral spectral multipliers, see Sections 3 and 5. One considers auxiliary nonzero compactly supported function $\eta \in C_c(a,b)$. Then for some Sobolev space $W^s_q(\mathbb{R})$ one can define a "local Sobolev norm" by the formula

$$\|F\|_{LW^s_q} = \sup_{t>0} \|\eta(\cdot)\, F(t\cdot)\|_{W^s_q(\mathbb{R})},$$

where $\|F\|_{W^s_q(\mathbb{R})} = \|(1-\frac{d^2}{dx^2})^{s/2}F\|_q$. Up to a constant this definition does not depend on a choice of the auxiliary function η as long as it is a non zero function. In singular spectral multipliers one wants to obtain estimates of $L^p \to L^p$ or weak (p,p) norm of the operator $F(L)$ in terms of the norm $\|F\|_{LW^s_q}$. In principle one expects that compact spectral multipliers imply singular spectral multipliers even though one can construct examples where $\sup_{t>0}\|\eta(L)F(tL)\|_{p\to p} \le C < \infty$ but $F(L)$ is unbounded, see well-known counterexample of Littman, McCarthy and Rivière [37].

• The most essential point of spectral multiplier theorems and Bochner Riesz analysis is an investigation of of Plancherel or restriction type estimates. Such estimates are essentially required to obtain compactly supported spectral multiplier. They originate from classical Fourier analysis where one considers so-called restriction problem: *describe all pairs of exponent* (p,q) *such that the restriction operator*

$$R_\lambda(f)(\omega) = \hat{f}(\lambda\omega)$$

is bounded from $L^p(\mathbb{R}^n) \to L^q(\mathbf{S}^{n-1})$. Here $\hat{f}$ is the Fourier transform of f and $\omega \in \mathbf{S}^{n-1}$ is a point on the unit sphere. It is not difficult to notice that if $E_{\sqrt{\Delta}}$ is the spectral resolution for the standard

Laplace operator, then

$$dE_{\sqrt{\Delta}}(\lambda) = \frac{\lambda^{n-1}}{(2\pi)^n} R_\lambda^* R_\lambda$$

(compare [28]). Now if one knows (as it is the case in Stein-Tomas restriction estimates, see [50]) that the restriction operator is bounded for some pair $(p, 2)$, then by T^*T trick it follows that $\frac{d}{d\lambda} E_{\sqrt{\Delta}}(\lambda)$ extends to a bounded operator acting form space $L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$, where p' is conjugate exponent of p $(1/p + 1/p' = 1)$. Note that these estimates can be expressed purely in terms of spectral resolutions of self-adjoint operator L. Motivated by the example of the standard Laplace operator we introduce below condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,2,m}})$. One of most significant part of spectral multiplier results and Bochner-Riesz analysis is to prove estimates of this type for some class of differential operators. Remarkable results of this type were obtained in [28]. Some other examples are described in [11]. In the case when $p = 1$ the problem quite often simplifies because of existence of underlying Plancherel measure and for example for homogeneous operators efficient restriction estimates type results are automatically true, see [22]. To illustrate our abstract spectral multiplier results, we describe some well known examples of operators which satisfy $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,2,m}})$ in Section 6. In Section 6.3 we describe new restriction estimates of this type.

In this paper we are mainly focus on proving that appropriate restriction type estimates imply sharp compactly supported spectral multiplier results and that the singular integral version follows from compactly supported spectral multipliers for abstract self-adjoint operators for which the corresponding heat kernels satisfy m-th order Gaussian bounds. We also discuss operators, which satisfy generalized Gaussian estimates in the sense of Blunck and Kunstmann, see e.g. [7]. The case $p = 1$ for such theory was comprehensively discussed in [22]. However if $p \neq 1$ the problem requires essentially new approach.

Under assumption that L satisfies finite speed propagation property, see [15, 44], similar results were considered in [11]. However there are many interesting examples of whole significant classes of operators which do not satisfy finite speed propagation property for the wave equation but satisfy m-th order Gaussian bound. For example m-th order differential operators or Laplace like operators defined on fractals, see for example [2, 47, 48]. The results obtained in this paper can be applied to these operators. Finite speed propagation property is equivalent to the second order Gaussian bounds, see [15, 44], so our paper can be regarded as the generalization of [11]. In particular our results apply to all examples discussed there. Note however that we are not able to obtain endpoint results in the current setting.

Our proof that compactly (dyadically) supported spectral multipliers imply singular integral multipliers is inspired by the work of Seeger and Sogge [41, 42]. However, there is no assumption on the regularity in variables x and y on the kernels $p_t(x, y)$ of the semigroup e^{-tL}, thus techniques of Calderón–Zygmund theory ([41, 42]) are not applicable. The lacking of smoothness of the kernel was indeed the main obstacle and it was overcome by using an approach to singular integral theory initiated by [29] and developed in [21, 1]. In this approach to obtain additional cancellation instead of subtracting some average of a function one subtracts appropriate multiplier of the operator L applied to the considered function.

In the sequel we always assume that considered ambient space is a metric measure space (X, d, μ) with metric d and Borel measure μ. We denote by $B(x, r) = \{y \in X,\ d(x, y) < r\}$ the open ball with centre $x \in X$ and radius $r > 0$. We also assume that the space X is homogeneous that is it satisfies the doubling condition. It allows us to consider the homogeneous dimension of the space X. To be more precise we put $V(x, r) = \mu(B(x, r))$ the volume of $B(x, r)$ and we say that (X, d, μ) satisfies the doubling property (see Chapter 3, [14]) if there exists a constant $C > 0$ such

that

$$V(x, 2r) \le CV(x, r) \quad \forall\, r > 0,\ x \in X. \tag{1.2}$$

If this is the case, there exist C, n such that for all $\lambda \ge 1$ and $x \in X$

$$V(x, \lambda r) \le C\lambda^n V(x, r). \tag{1.3}$$

In the sequel we want to consider n as small as possible. Note that in general one cannot take infimum over such exponents n in (1.3). In the Euclidean space with Lebesgue measure, n corresponds to the dimension of the space. In our results critical index is always expressed in terms of homogeneous dimension n. Usually existence of $s > n(1/2 - 1/p)$ derivatives of function F is a sharp optimal condition in most of spectral multiplier results, both compact and singular. However there is a subtle but of huge significance difference between existence of this number of derivatives in $L^2(\mathbb{R})$ versus $L^\infty(\mathbb{R})$. Improvement of the results from L^∞ to L^2 always requires some form of restriction or Plancherel type of estimates. This L^2 spectral multiplier results essentially corresponds to calculation of critical exponent in Bochner-Riesz means analysis and is related to Bochner-Riesz conjecture. Obtaining sharp results in this context is regarded as one of most crucial tasks in harmonic analysis.

2. Preliminaries

We commence with describing our notation and basic assumptions. We often just use B instead of $B(x, r)$. Given $\lambda > 0$, we write λB for the λ-dilated ball which is the ball with the same centre as B and radius λr. For $1 \le p \le +\infty$, we denote the norm of a function $f \in L^p(X)$ by $\|f\|_p$, by $\langle .,. \rangle$ the scalar product of $L^2(X)$, and if T is a bounded linear operator from $L^p(X)$ to $L^q(X)$, $1 \le p,\ q \le +\infty$, we write $\|T\|_{p\to q}$ for the operator norm of T. Given a subset $E \subseteq X$, we denote by χ_E the characteristic function of E and set

$$P_E f(x) = \chi_E(x) f(x).$$

For every $B = B(x_B, r_B)$, set $A(x_B, r_B, 0) = B$ and

$$A(x_B, r_B, j) = B(x_B, (j+1)r_B) \backslash B(x_B, jr_B), \qquad j = 1, 2, \ldots$$

For a given function $F : \mathbb{R} \to \mathbb{C}$ and $R > 0$, we define the function $\delta_R F : \mathbb{R} \to \mathbb{C}$ by putting $\delta_R F(x) = F(Rx)$. Given $p \in [1, \infty]$, the conjugate exponent p' is defined by $1/p + 1/p' = 1$. We will also use the Hardy-Littlewood maximal operator $\mathcal{M}f$ which is defined by

$$\mathcal{M}f(x) = \sup_{B \ni x} \frac{1}{V(B)} \int_B |f(y)| d\mu(y),$$

where the *sup* is taken over all balls B containing x.

2.1. Generalized Gaussian estimates and Davies-Gaffney estimates. We now described the notion of the Generalized Gaussian estimates introduced by Blunck and Kunstmann, see [5, 6, 7]. Consider a non-negative self-adjoint operator L and numbers $m \ge 2$ and $1 \le p \le 2 \le q \le \infty$ with $p < q$. We say that the semigroup generated by L, e^{-tL} satisfies *generalized Gaussian* (p, q)-*estimates of order m*, if there exist constants $C, c > 0$ such that for all $t > 0$, and all $x, y \in X$,

$$\left\| P_{B(x,t^{1/m})} e^{-tL} P_{B(y,t^{1/m})} \right\|_{p\to q} \le CV(x, t^{1/m})^{-(\frac{1}{p}-\frac{1}{q})} \exp\Big(- c\Big(\tfrac{d(x,y)}{t^{1/m}}\Big)^{\frac{m}{m-1}} \Big). \tag{$\mathrm{GGE_{p,q,m}}$}$$

Note that condition ($\mathrm{GGE_{p,q,m}}$) for the special case $(p, q) = (1, \infty)$ is equivalent to m-th order Gaussian estimates (see Proposition 2.9, [5]). This means that the semigroup e^{-tL} has integral

kernels $p_t(x,y)$ satisfying the following estimates

$$(\mathrm{GE_m}) \qquad |p_t(x,y)| \le \frac{C}{V(x,t^{1/m})}\exp\Big(-c\Big(\frac{d^m(x,y)}{t}\Big)^{\frac{1}{m-1}}\Big) \text{ for } x,y\in X,\ t>0.$$

There are numbers of operators which satisfy generalized Gaussian estimates and, among them, there exist many for which classical Gaussian estimates $(\mathrm{GE_m})$ fail. This happens, e.g., for Schrödinger operators with rough potentials [40], second order elliptic operators with rough lower order terms [36], or higher order elliptic operators with bounded measurable coefficients [18].

The following result originally stated in [51, Lemma 2.6] (see also [3, Theorem 2.1]) shows that generalized Gaussian estimates can be extended from real times $t>0$ to complex times $z\in\mathbb{C}$ with $\mathrm{Re} z>0$.

Lemma 2.1. *Let $m\ge 2$ and $1\le p\le 2\le q\le\infty$, and L be a non-negative self-adjoint operator on $L^2(X)$. Assume that there exist constants $C,c>0$ such that for all $t>0$, and all $x,y\in X$,*

$$\big\|P_{B(x,t^{1/m})}e^{-tL}P_{B(y,t^{1/m})}\big\|_{p\to q} \le CV(x,t^{1/m})^{-(\frac1p-\frac1q)}\exp\Big(-c\Big(\frac{d(x,y)}{t^{1/m}}\Big)^{\frac{m}{m-1}}\Big).$$

Let $r_z=(\mathrm{Re}\, z)^{\frac1m-1}|z|$ for each $z\in\mathbb{C}$ with $\mathrm{Re}\, z>0$.

(i) *There exist two positive constants C' and c' such that for all $r>0$, $x\in X$, and $z\in\mathbb{C}$ with $\mathrm{Re}\, z>0$*

$$\begin{aligned}&\big\|P_{B(x,r)}e^{-zL}P_{B(y,r)}\big\|_{p\to q}\\ &\quad\le\ C'V(x,r)^{-(\frac1p-\frac1q)}\Big(1+\frac{r}{r_z}\Big)^{n(\frac1p-\frac1q)}\Big(\frac{|z|}{\mathrm{Re}\, z}\Big)^{n(\frac1p-\frac1q)}\exp\Big(-c'\Big(\frac{d(x,y)}{r_z}\Big)^{\frac{m}{m-1}}\Big).\end{aligned}$$

(ii) *There exist two positive constants C'' and c'' such that for all $r>0$, $x\in X$, $k\in\mathbb{N}$ and $z\in\mathbb{C}$ with $\mathrm{Re}\, z>0$*

$$\begin{aligned}&\big\|P_{B(x,r)}e^{-zL}P_{A(x,r,k)}\big\|_{p\to q}\\ &\quad\le\ C''V(x,r)^{-(\frac1p-\frac1q)}\Big(1+\frac{r}{r_z}\Big)^{n(\frac1p-\frac1q)}\Big(\frac{|z|}{\mathrm{Re}\, z}\Big)^{n(\frac1p-\frac1q)}k^n\exp\Big(-c''\Big(\frac{r}{r_z}k\Big)^{\frac{m}{m-1}}\Big).\end{aligned}$$

Proof. For the detailed proof we refer readers to [51]. Here we only want to mention that the proof of Lemma 2.1 relies on the Phragmén-Lindelöf theorem. □

Next suppose that $m\ge 2$. We say that the semigroup e^{-tL} generated by non-negative self-adjoint operator L satisfies *m-th order Davies-Gaffney estimates*, if there exist constants $C,c>0$ such that for all $t>0$, and all $x,y\in X$,

$$(\mathrm{DG_m}) \qquad \big\|P_{B(x,t^{1/m})}e^{-tL}P_{B(y,t^{1/m})}\big\|_{2\to 2}\le C\exp\Big(-c\Big(\frac{d(x,y)}{t^{1/m}}\Big)^{\frac{m}{m-1}}\Big).$$

Note that if condition $(\mathrm{GGE_{p,q,m}})$ holds for for some $1\le p\le 2\le q\le\infty$ with $p<q$, then the semigroup e^{-tL} satisfies estimate $(\mathrm{DG_m})$.

The following lemma describes a useful consequence of m-order Davies-Gaffney estimates.

Lemma 2.2. *Let $m\ge 2$ and let e^{-tL} be a semigroup generated by a non-negative, self-adjoint operator L acting on $L^2(X)$ satisfying Davies-Gaffney estimates $(\mathrm{DG_m})$. Then for every $M>0$, there exists a constant $C=C(M)$ such that for every $j=2,3,\dots$*

$$(2.1) \qquad \big\|P_BF(\sqrt[m]{L})P_{A(x_B,r_B,j)}\big\|_{2\to 2}\le Cj^{-M}(Rr_B)^{-(M+n)}\|\delta_RF\|_{W_2^{M+n+1}}$$

for all balls $B \subseteq X$, and all Borel functions F such that supp $F \subseteq [R/4, R]$.

Proof. Let $G(\lambda) = \delta_R F(\sqrt[m]{\lambda})e^{\lambda}$. In virtue of the Fourier inversion formula

$$G(L/R^m)e^{-L/R^m} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{(i\tau-1)R^{-m}L}\hat{G}(\tau)d\tau$$

so

$$\|P_B F(\sqrt[m]{L})P_{A(x_B,r_B,j)}\|_{2\to 2} \le \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{G}(\tau)|\, \|P_B e^{(i\tau-1)R^{-m}L}P_{A(x_B,r_B,j)}\|_{2\to 2} d\tau.$$

By (ii) of Lemma 2.1 (with $r_z = \sqrt{1+\tau^2}/R$),

$$\begin{aligned}
\|P_B e^{(i\tau-1)R^{-m}L}P_{A(x_B,r_B,j)}\|_{2\to 2} &\le C j^n \exp\Big(-c\Big(\frac{Rjr_B}{\sqrt{1+\tau^2}}\Big)^{\frac{m}{m-1}}\Big)\\
&\le C_M j^n\Big(\frac{Rjr_B}{\sqrt{1+\tau^2}}\Big)^{-M-n}\\
&\le C j^{-M}(1+\tau^2)^{\frac{M+n}{2}}(Rr_B)^{-(M+n)}.
\end{aligned}$$

Therefore (compare [22, (4.4)])

$$\begin{aligned}
&\|P_B F(\sqrt[m]{L})P_{A(x_B,r_B,j)}\|_{2\to 2}\\
&\le C j^{-M}(Rr_B)^{-(M+n)}\int_{\mathbb{R}}|\hat{G}(\tau)|(1+\tau^2)^{\frac{M+n}{2}}d\tau\\
&\le C j^{-M}(Rr_B)^{-(M+n)}\Big(\int_{\mathbb{R}}|\hat{G}(\tau)|^2(1+\tau^2)^{M+n+1}d\tau\Big)^{1/2}\Big(\int_{\mathbb{R}}(1+\tau^2)^{-1}d\tau\Big)^{1/2}\\
&\le C j^{-M}(Rr_B)^{-(M+n)}\|G\|_{W_2^{M+n+1}}.
\end{aligned}$$

However, supp $F \subseteq [R/4, R]$ and supp $\delta_R F \in [1/4, 1]$ so

$$\|G\|_{W_2^{M+n+1}} \le C\|\delta_R F\|_{W_2^{M+n+1}}.$$

This ends the proof of Lemma 2.1. □

2.2. The Stein-Tomas restriction type condition. Consider a non-negative self-adjoint operator L and numbers p and q such that $1 \le p < 2$ and $1 \le q \le \infty$. We say that L satisfies the *Stein-Tomas restriction type condition* if: for any $R > 0$ and all Borel functions F such that supp $F \subset [0, R]$,

$$(\mathrm{ST}^{q}_{p,2,m}) \qquad \big\|F(\sqrt[m]{L})P_{B(x,r)}\big\|_{p\to 2} \le CV(x,r)^{\frac12-\frac1p}(Rr)^{n(\frac1p-\frac12)}\big\|\delta_R F\big\|_q$$

for all $x \in X$ and all $r \ge 1/R$.

An interesting alternative approach to restriction type estimates is investigated by Kunnstman and Uhm in [35, 51], see (4.3) of [35] and the Plancharel condition (5.30) of [51]. Let us point out that estimate $(\mathrm{ST}^{2}_{p,2,m})$ implies sharp Bochner Riesz results for all p.

Note that if condition $(\mathrm{ST}^{q}_{p,2,m})$ holds for some $q \in [1, \infty)$, then $(\mathrm{ST}^{\tilde q}_{p,2,m})$ holds for all $\tilde q \ge q$ including the case $\tilde q = \infty$.

The next lemma shows that if $q = \infty$ then condition $(\mathrm{ST}^{\infty}_{p,2,m})$ follows form the standard elliptic estimates.

Proposition 2.3. *Suppose that (X, d, μ) satisfies property (1.2) and (1.3). Let $1 \le p < 2$ and $N > n(1/p - 1/2)$. Then $(\mathrm{ST}^{\infty}_{p,2,m})$ is equivalent to each of the following conditions:*

(a) *For all* $x \in X$ *and* $r \ge t > 0$

$$\left\|e^{-t^m L} P_{B(x,r)}\right\|_{p\to 2} \le C V(x,r)^{\frac{1}{2}-\frac{1}{p}} \Big(\frac{r}{t}\Big)^{n(\frac{1}{p}-\frac{1}{2})}. \tag{$\mathrm{G_{p,2,m}}$}$$

(b) *For all* $x \in X$ *and* $r \ge t > 0$

$$\left\|(I + t\sqrt[m]{L})^{-N} P_{B(x,r)}\right\|_{p\to 2} \le C V(x,r)^{\frac{1}{2}-\frac{1}{p}} \left(\frac{r}{t}\right)^{n(\frac{1}{p}-\frac{1}{2})}. \tag{$\mathrm{E_{p,2,m}}$}$$

Proof. The proof is originally given in [11] only second-order operators. However, with some minor modifications, the proof can be adapted to the mth-order version, and we omit the detail here. □

The following lemma is a standard known result in the theory of spectral multipliers of non-negative selfadjoint operators and it is a version of [11, Lemma 4.5] adjusted to the setting of m-order operators so we use the same notation.

Lemma 2.4. *Suppose that operator* L *is a non-negative self-adjoint operator* L *on* $L^2(X)$ *satisfying Davies-Gaffney estimates* ($\mathrm{DG_m}$) *and condition* ($\mathrm{G_{p_0,2,m}}$) *for some* $1 \le p_0 < 2$.

(a) *Assume in addition that* F *is an even bounded Borel function such that*

$$\sup_{t>0} \|\eta\delta_t F\|_{C^k} < \infty$$

for some integer $k > n/2 + 1$ *and some non-trivial function* $\eta \in C_c^\infty(0,\infty)$. *Then the operator* $F(\sqrt[m]{L})$ *is bounded on* $L^p(X)$ *for all* $p_0 < p < p_0'$.

(b) *Assume in addition that* ψ *be a function in* $\mathscr{S}(\mathbb{R})$ *such that* $\psi(0) = 0$. *Define the quadratic functional for* $f \in L^2(X)$

$$\mathcal{G}_L(f)(x) = \Big(\sum_{j\in\mathbb{Z}} |\psi(2^j \sqrt[m]{L}) f|^2\Big)^{1/2}.$$

Then $\mathcal{G}_L$ *is bounded on* $L^p(X)$ *for all* $p_0 < p < p_0'$.

Proof. It follows from ($\mathrm{G_{p_0,2,m}}$) that

$$\|P_{B(x,t^{1/m})} e^{-tL} P_{B(y,t^{1/m})}\|_{p_0\to 2} \le C V(x,t^{1/m})^{\frac{1}{2}-\frac{1}{p_0}}. \tag{2.2}$$

Let $r \in (p_0, 2)$. By (2.2) and Davies-Gaffney estimates ($\mathrm{DG_m}$), the Riesz-Thorin interpolation theorem gives the following $L^r - L^2$ off-diagonal estimate

$$\|P_{B(x,t^{1/m})} e^{-tL} P_{B(y,t^{1/m})}\|_{r\to 2} \le C V(x,t^{1/m})^{\frac{1}{2}-\frac{1}{r}} \exp\Big(-c\Big(\frac{d(x,y)}{t^{1/m}}\Big)^{\frac{m}{m-1}}\Big) \tag{2.3}$$

for all $x, y \in X$ and $t > 0$. Assertion (a) then follows from [4]. The latter off-diagonal estimate implies that L has a bounded holomorphic functional calculus on L^p for $p_0 < p < p_0'$ (see [4]). It is known that the holomorphic functional calculus implies the quadratic estimate of assertion (b) (see [16, 38]). □

3. A criterion for L^p boundedness of spectral multipliers

In this section, we shall state and prove a criterion for L^p boundedness of spectral multipliers. In many cases, this theorem allows us to reduce the proof of the L^p-boundedness of general multiplier operator $F(L)$ to obtaining estimates for operators corresponding to dyadically supported functions. Then in the next section, we will show that sharp results for spectral multipliers with dyadic support follows from restriction type conditions.

In what follows, we fix a non-zero C^∞ bump function on $\mathbb{R}$ such that

$$\mathrm{supp}\phi \subseteq (\frac{1}{2}, 2) \ \text{ and } \ \sum_{\ell\in\mathbb{Z}} \phi(2^{-\ell}\lambda) = 1 \quad \text{for all } \lambda > 0 \tag{3.1}$$

and set $\phi_\ell(\lambda) = \phi(\lambda/2^\ell)$.

The aim of this section is to prove the following result.

Theorem 3.1. *Let L be a non-negative self-adjoint operator L on $L^2(X)$ satisfying Davies-Gaffney estimates* $(\mathrm{DG_m})$ *and condition* $(\mathrm{G_{p_0,2,m}})$ *for some $1 \le p_0 < 2$. Let F be a bounded Borel function such that for $p \in (p_0, p_0')$,*

$$\sup_{t>0} \|(\phi\delta_t F)(\sqrt[m]{L})\|_{p\to p} + \sup_{t>0} \|(\phi\delta_t F)(\sqrt[m]{L})\|_{2\to 2} \le A \tag{3.2}$$

holds. Then for every $M > n/2 + 1$, there exists a constant $C > 0$ such that

$$\|F(\sqrt[m]{L})\|_{p\to p} \le CA\Big\{\log\Big(2 + \frac{\sup_{t>0}\|\phi\delta_t F\|_{W_2^{M+n+1}}}{A}\Big)\Big\}^{|\frac{1}{p}-\frac{1}{2}|}.$$

The proof of Theorem 3.1 is inspired by ideas developed in [1, 21, 29, 41, 42].

Let us introduce some tools needed in the proof. Let T be a sublinear operator which is bounded on $L^2(X)$. Let $\{A_r\}_{r>0}$ be a family of linear operators acting on $L^2(X)$. For $f \in L^2(X)$, we follow [1] to define

$$\mathcal{M}^{\#}_{T,A} f(x) = \sup_{B\ni x} \Big(\frac{1}{V(B)}\int_B \big|T(I - A_{r_B})f\big|^2 d\mu\Big)^{1/2},$$

where the supremum is taken over all balls B in X containing x, and r_B is the radius of B.

Proposition 3.2. *Suppose that T is a sublinear operator which is bounded on $L^2(X)$ and that $q \in (2, \infty]$. Assume that $\{A_r\}_{r>0}$ is a family of linear operators acting on $L^2(X)$ and that*

$$\Big(\frac{1}{V(B)}\int_B |TA_{r_B} f(y)|^q d\mu(y)\Big)^{1/q} \le C(\mathcal{M}(|Tf|^2)^{1/2}(x) \tag{3.3}$$

for all $f \in L^2(X)$, all $x \in X$ and all balls $B \ni x$, r_B being the radius of B.

Then for $0 < p < q$, there exists C_p such that

$$\big\|(\mathcal{M}(|Tf|^2))^{1/2}\|_p \le C_p(\|\mathcal{M}^{\#}_{T,A} f)\|_p + \|f\|_p) \tag{3.4}$$

for every $f \in L^2(X)$ for which the left-hand side is finite (if $\mu(X) = \infty$, the term $C_p\|f\|_p$ can be omitted in the right-hand side of (3.4).

Proof. For the proof of Proposition 3.2, we refer readers to [1, Lemma 2.3]. □

Proof of Theorem 3.1. Given a bounded Borel function F, we consider an operator T_F, given by

$$T_F f(x) = \Big\{\sum_{k\in\mathbb{Z}} |(\phi_k^2 F)(\sqrt[m]{L}) f(x)|^2\Big\}^{1/2}.$$

For this operator T_F, condition (3.3) always holds for every $2 < q < p_0'$ and $A_{r_B} = I - (I - e^{-r_B^m L})^K$ for every $K \in \mathbb{N}$. Indeed, in virtue of the formula

$$I - (I - e^{-r_B^m L})^K = \sum_{s=1}^{K} \binom{K}{s} (-1)^{s+1} e^{-s r_B^m L}$$

and the commutativity property $(\phi_k^2 F)(\sqrt[m]{L}) e^{-s r_B^m L} = e^{-s r_B^m L} (\phi_k^2 F)(\sqrt[m]{L})$, it is enough to show that for all $B \ni x$,

$$\Big(\frac{1}{V(B)} \int_B \Big(\sum_{k\in\mathbb{Z}} \big| e^{-s r_B^m L} (\phi_k^2 F)(\sqrt[m]{L}) f(y) \big|^2 \Big)^{q/2} d\mu(y) \Big)^{1/q} \le C(\mathcal{M}(|T_F f|^2)^{1/2}(x). \tag{3.5}$$

To prove (3.5), we observe that hypothesis ($\mathrm{DG_m}$) and ($\mathrm{G_{p_0,2,m}}$) imply ($\mathrm{GGE_{q',2,m}}$). By duality, ($\mathrm{GGE_{2,q,m}}$) holds. By Minkowski's inequality, (ii) of Lemma 2.1, conditions (1.2) and (1.3) for every $s = 1, 2, \dots, K$ and every $B \ni x$, the left hand side of (3.5) is less than

$$\begin{aligned}
& V(B)^{-1/q} \sum_{j=0}^{\infty} \Big\{ \sum_{k\in\mathbb{Z}} (\| P_B e^{-s r_B^m L} P_{A(x_B, r_B, j)} (\phi_k^2 F)(\sqrt[m]{L}) f \|_q)^2 \Big\}^{1/2} \\
\le \;& V(B)^{-1/q} \sum_{j=0}^{\infty} \| P_B e^{-s r_B^m L} P_{A(x_B, r_B, j)} \|_{2\to q} \Big\{ \sum_{k\in\mathbb{Z}} \| (\phi_k^2 F)(\sqrt[m]{L}) f \|^2_{L^2(A(x_B, r_B, j))} \Big\}^{1/2} \\
\le \;& C \sum_{j=0}^{\infty} \Big(\frac{V((j+1)B)}{V(B)} \Big)^{1/2} e^{-c_s j^{m/(m-1)}} \Big\{ \frac{1}{V((j+1)B)} \int_{(j+1)B} \sum_{k\in\mathbb{Z}} \big| (\phi_k^2 F)(\sqrt[m]{L}) f(y) \big|^2 d\mu(y) \Big\}^{1/2} \\
\le \;& C \sum_{j=0}^{\infty} e^{-c_s j^{m/(m-1)}} (1+j)^{n/2} (\mathcal{M}(|T_F f|^2)^{1/2}(x) \\
\le \;& C(\mathcal{M}(|T_F f|^2)^{1/2}(x).
\end{aligned}$$

The above estimates yield (3.5).

Define, for every $K \in \mathbb{N}$ and every $f \in L^2(X)$,

$$\mathcal{M}^{\#}_{T_F, L, K} f(x) = \sup_{B \ni x} \Big(\frac{1}{V(B)} \int_B \big| T_F (I - e^{-r_B^m L})^K f(y) \big|^2 d\mu(y) \Big)^{1/2}, \tag{3.6}$$

where the supremum is taken over all balls B in X containing x, and r_B is the radius of B. Note that by duality it suffices to prove Theorem 3.1 for $2 < p < p_0'$. We shall show that if K is large enough, then

$$\big\| \mathcal{M}^{\#}_{T_F, L, K} f \big\|_p \le C A N^{\frac{1}{2} - \frac{1}{p}} \|f\|_p, \tag{3.7}$$

where A is given in (3.2), and

$$N = \log\Big(2 + \frac{\sup_{t>0} \|\phi \delta_t F\|_{W_2^{M+n+1}}}{A} \Big). \tag{3.8}$$

Once we show estimates (3.7) and (3.8), it follows from (b) of Lemma 2.4 and Proposition 3.2 (with some $p < q < p_0'$) that

$$\begin{aligned}
\| F(\sqrt[m]{L}) f \|_p &\le C \| T_F f \|_p \\
&\le C \| (\mathcal{M}(|T_F f|^2))^{1/2} \|_p \\
&\le C_p (\| \mathcal{M}^{\#}_{T_F, L, K} f \|_p + \| f \|_p)
\end{aligned}$$

$$\leq \ CAN^{\frac{1}{2}-\frac{1}{p}}\|f\|_p,$$

and this concludes the proof of Theorem 3.1.

Therefore it suffices to prove (3.7). By Minkowski's inequality

$$\mathcal{M}^{\#}_{T_F,L,K} f(x) \leq \mathscr{E}_1(f)(x) + \mathscr{E}_2(f)(x),$$

where

$$\mathscr{E}_1(f)(x) = \sup_{B\ni x}\Big(\frac{1}{V(B)}\int_B \sum_{|k+\log_2 r_B|\leq N}\Big|(I-e^{-r_B^m L})^K(\phi_k^2 F)(\sqrt[m]{L})f(y)\Big|^2 d\mu(y)\Big)^{1/2}$$

and

$$\mathscr{E}_2(f)(x) = \sup_{B\ni x}\Big(\frac{1}{V(B)}\int_B \sum_{|k+\log_2 r_B|> N}\Big|(I-e^{-r_B^m L})^K(\phi_k^2 F)(\sqrt[m]{L})f(y)\Big|^2 d\mu(y)\Big)^{1/2}.$$

To prove estimate (3.7) it is enough to show that

$$\|\mathscr{E}_1(f)\|_p \leq CN^{\frac{1}{2}-\frac{1}{p}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\Big|(\phi_k^2F)(\sqrt[m]{L})f\Big|^p\Big)^{1/p}\Big\|_p \tag{3.9}$$

and

$$\|\mathscr{E}_2(f)\|_p \leq CA\Big\|\Big(\sum_{k\in\mathbb{Z}}\Big|\phi_k(\sqrt[m]{L})f\Big|^2\Big)^{1/2}\Big\|_p. \tag{3.10}$$

Indeed, by (3.9) and (3.2),

$$\begin{aligned}
\Big\|\Big(\sum_{k\in\mathbb{Z}}\Big|(\phi_k^2F)(\sqrt[m]{L})f\Big|^p\Big)^{1/p}\Big\|_p^p &\leq \sum_k \Big\|(\phi_k^2F)(\sqrt[m]{L})f\Big\|_p^p\\
&\leq \sup_{k\in\mathbb{Z}}\Big\|(\phi_kF)(\sqrt[m]{L})\Big\|_{p\to p}^p \sum_k \Big\|\phi_k(\sqrt[m]{L})f\Big\|_p^p\\
&\leq A^p\Big\|\Big(\sum_{k\in\mathbb{Z}}|\phi_k(\sqrt[m]{L})f|^2\Big)^{1/2}\Big\|_p^p\\
&\leq CA^p\|f\|_p^p.
\end{aligned}$$

Noq by (3.10) and Lemma 2.4,

$$\|\mathscr{E}_2(f)\|_p \leq CA\|f\|_p.$$

These estimates imply (3.7).

It remains to prove claims (3.9) and (3.10).

Proof of (3.9) Similarly as in the proof of (3.5), (DG$_{\rm m}$) and (ii) of Lemma 2.1 yields

$$\begin{aligned}
&\mathscr{E}_1(f)(x)\\
&\leq C_K\sum_{s=1}^K\sum_{j=0}^\infty (1+j)^{\frac{n}{2}}e^{-c_s j^{\frac{m}{m-1}}}\Big\{\sup_{B\ni x}\frac{1}{V((j+1)B)}\int_{(j+1)B}\sum_{|k+\log_2 r_B|\leq N}\Big|(\phi_k^2F)(\sqrt[m]{L})f\Big|^2 d\mu\Big\}^{1/2}\\
&\leq C_K N^{\frac{1}{2}-\frac{1}{p}}\sum_{s=1}^K\sum_{j=0}^\infty (1+j)^{\frac{n}{2}}e^{-c_s j^{\frac{m}{m-1}}}\sup_{B\ni x}\Big\{\frac{1}{V((j+1)B)}\int_{(j+1)B}\Big(\sum_{k\in\mathbb{Z}}\Big|(\phi_k^2F)(\sqrt[m]{L})f\Big|^p\Big)^{2/p} d\mu\Big\}^{1/2}\\
&\leq C_K N^{\frac{1}{2}-\frac{1}{p}}\Big(\mathcal{M}\Big(\Big\{\sum_{k\in\mathbb{Z}}\Big|(\phi_k^2F)(\sqrt[m]{L})f\Big|^p\Big\}^{2/p}\Big)\Big)^{1/2}(x).
\end{aligned}$$

Then by $L^{p/2}$-boundedness of $\mathcal{M}$

$$\|\mathscr{E}_1(f)\|_p \ \leq\ CN^{\frac{1}{2}-\frac{1}{p}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\Big|(\phi_k^2F)(\sqrt[m]{L})f\Big|^p\Big)^{1/p}\Big\|_p.$$

This shows (3.9).

Proof of (3.10). We need a further decomposition of $\mathscr{E}_2(f)$. Note that

$$\mathscr{E}_2(f)(x) \le \mathscr{E}_{21}(f)(x) + \mathscr{E}_{22}(f)(x),$$

where

$$\mathscr{E}_{21}(f)(x) = \sup_{B\ni x}\Big(\frac{1}{V(B)}\int_B\sum_{k\in\mathbb{Z}}\Big|(1-e^{-r_B^mL})^K(\phi_kF)(\sqrt[m]{L})(P_{2B}\phi_k(\sqrt[m]{L})f)(y)\Big|^2d\mu(y)\Big)^{1/2}$$

and

$$\mathscr{E}_{22}(f)(x) = \sup_{B\ni x}\Big(\frac{1}{V(B)}\int_B\sum_{|k+\log_2 r_B|>N}\Big|(I-e^{-r_B^mL})^K(\phi_kF)(\sqrt[m]{L})(P_{X\backslash 2B}\phi_k(\sqrt[m]{L})f)(y)\Big|^2d\mu(y)\Big)^{1/2}.$$

We first estimate the term $\mathscr{E}_{21}(f)$. By (3.2)

$$\begin{aligned}
&\mathscr{E}_{21}(f)(x)\\
&\le \sup_{B\ni x}\Big(\frac{1}{V(B)}\sum_{k\in\mathbb{Z}}\Big\|(1-e^{-r_B^mL})^K(\phi_kF)(\sqrt[m]{L})(P_{2B}\phi_k(\sqrt[m]{L})f)\Big\|_2^2\Big)^{1/2}\\
&\le C\sup_{r_B>0}\sup_{k\in\mathbb{Z}}\Big\|(1-e^{-r_B^mL})^K(\phi_kF)(\sqrt[m]{L})\Big\|_{2\to2}\sup_{B\ni x}\Big(\frac{1}{V(2B)}\int_{2B}\sum_{k\in\mathbb{Z}}|\phi_k(\sqrt[m]{L})f(y)|^2d\mu(y)\Big)^{1/2}\\
&\le CA(\mathcal{M}(\sum_{k\in\mathbb{Z}}|\phi_k(\sqrt[m]{L})f|^2))^{1/2}(x).
\end{aligned}$$

By $L^{p/2}$-boundedness of $\mathcal{M}$

$$\begin{aligned}
\|\mathscr{E}_{21}(f)\|_p &\le CA\Big\|(\mathcal{M}(\sum_{k\in\mathbb{Z}}|\phi_k(\sqrt[m]{L})f|^2))^{1/2}\Big\|_p\\
&\le CA\Big\|(\sum_{k\in\mathbb{Z}}\Big|\phi_k(\sqrt[m]{L})f\Big|^2)^{1/2}\Big\|_p.
\end{aligned}$$

Next, we consider the term $\mathscr{E}_{22}(f)$. Observe that

$$\mathscr{E}_{22}(f)(x) \le \sup_{B\ni x}\sum_{j=2}^{\infty}\sum_{|k+\log_2 r_B|>N}V(B)^{-1/2}\Big\|P_B(1-e^{-r_B^mL})^K(\phi_kF)(\sqrt[m]{L})(P_{A(x_B,r_B,j)}\phi_k(\sqrt[m]{L})f)\Big\|_2.$$

By conditions (1.2) and (1.3),

$$\begin{aligned}
&\Big\|P_B(1-e^{-r_B^mL})^K(\phi_kF)(\sqrt[m]{L})(P_{A(x_B,r_B,j)}\phi_k(\sqrt[m]{L})f)\Big\|_2\\
&\quad\le Cj^{n/2}V(B)^{1/2}\Big\|P_B(1-e^{-r_B^mL})^K(\phi_kF)(\sqrt[m]{L})P_{A(x_B,r_B,j)}\Big\|_{2\to2}\times\\
&\qquad\qquad\times\Big(\frac{1}{V((j+1)B)}\int_{(j+1)B}|\phi_k(\sqrt[m]{L})f(y)|^2d\mu(y)\Big)^{1/2}.
\end{aligned}$$

To continue, we note that the function $(1-e^{-r_B^m\lambda^m})^K\phi_k(\lambda)F(\lambda)$ is supported in $[2^{k-1},\ 2^{k+1}]$. Now, if k is an integer greater than $M+n+1$, then for Sobolev space $W_2^{M+n+1}(\mathbb{R})$

$$\begin{aligned}
&\|\delta_{2^{k+1}}((1-e^{-r_B^m\lambda^m})^K\phi_k(\lambda)F(\lambda))\|_{W_2^{M+n+1}}\\
&\le \|(1-e^{-(2^{k+1}r_B\lambda)^m})^K\phi(\lambda)\delta_{2^{k+1}}F(\lambda)\|_{W_2^{M+n+1}}\\
&\le \|(1-e^{-(2^{k+1}r_B\lambda)^m})^K\|_{C^k([\frac12,1])}\|\phi\delta_{2^{k+1}}F\|_{W_2^{M+n+1}}
\end{aligned}$$

$$\leq C \min\{1,\ (2^k r_B)^{mK}\} \sup_{t>0} \|\phi\delta_t F\|_{W_2^{M+n+1}}.$$

By Lemma 2.2 for every $M > 0$

$$\begin{aligned}
&\left\| P_B(1-e^{-r_B^m L})^K (\phi_k F)(\sqrt[m]{L}) P_{A(x_B, r_B, j)} \right\|_{2\to 2} \\
&\leq C j^{-M} (2^k r_B)^{-M-n} \|\delta_{2^{k+1}}((1-e^{-r_B^m \lambda^m})^K \phi_k(\lambda) F(\lambda))\|_{W_2^{M+n+1}} \\
&\leq C j^{-M} \min\{(2^k r_B)^{-M-n},\ (2^k r_B)^{mK-M-n}\} \sup_{t>0} \|\phi\delta_t F\|_{W_2^{M+n+1}}.
\end{aligned}$$

Therefore, we sum a geometrical series to obtain that if $M > n/2+1$ and $mK - M - n > 1$ in (3.6), then

$$\begin{aligned}
&\mathscr{E}_{22}(f)(x) \leq C \sup_{t>0} \|\phi\delta_t F\|_{W_2^{M+n+1}} \sup_{B\ni x} \sum_{j=2}^{\infty} j^{n/2-M} \\
&\times \sum_{|k+\log_2 r_B|>N} \min\{(2^k r_B)^{-M-n},\ (2^k r_B)^{mK-M-n}\} \Big(\frac{1}{V((j+1)B)} \int_{(j+1)B} |\phi_k(\sqrt[m]{L}) f|^2 d\mu \Big)^{1/2} \\
&\leq C 2^{-N} \sup_{t>0} \|\phi\delta_t F\|_{W_2^{M+n+1}} \sum_{j=2}^{\infty} j^{n/2-M} \sup_{B\ni x} \Big(\frac{1}{V((j+1)B)} \int_{(j+1)B} \sum_{k\in\mathbb{Z}} |\phi_k(\sqrt[m]{L}) f|^2 d\mu \Big)^{1/2} \\
&\leq CA \Big(\mathcal{M} \Big(\sum_{k\in\mathbb{Z}} |\phi_k(\sqrt[m]{L}) f|^2 \Big) \Big)^{1/2} (x),
\end{aligned}$$

where we use the fact that by condition (3.8), $2^{-N} \sup_{t>0} \|\phi\delta_t F\|_{W_2^{M+n+1}} \leq CA$.

Again, by $L^{p/2}$-boundedness of $\mathcal{M}$

$$\|\mathscr{E}_{22}(f)\|_p \ \leq\ CA \Big\| \Big(\sum_{k\in\mathbb{Z}} |\phi_k(\sqrt[m]{L}) f|^2 \Big)^{1/2} \Big\|_p.$$

This shows (3.10) and completes the proof of Theorem 3.1. □

By a classical dyadic decomposition of F, we can write $F(\sqrt[m]{L})$ as the sum $\sum F_j(\sqrt[m]{L})$. Then we apply Theorem 3.1 to estimate $\|F_j(\sqrt[m]{L})\|_{r\to r}$. However, as mentioned in the introduction, this does not automatically imply that the operator $F(\sqrt[m]{L})$ acts boundedly on L^r. See [10, 41, 42] where this problem is discussed in the Euclidean case.

We shall now discuss a a criterion which guarantee boundedness of $F(L)$ under assumption that multipliers supported in dyadic intervals are uniformly bounded. In Section 4 we describe results concerning multipliers supported in dyadic intervals.

Theorem 3.3. *Let L be a non-negative self-adjoint operator L on $L^2(X)$ satisfying Davies-Gaffney estimates* $(\mathrm{DG_m})$ *and condition* $(\mathrm{G_{p_0,2,m}})$ *for some* $1 \leq p_0 < 2$*. Assume that for any bounded Borel function H such that supp* $H \subseteq [1/4, 4]$*, the following condition holds:*

$$\sup_{t>0} \|H(t\sqrt[m]{L})\|_{p\to p} \leq C\|H\|_{W_q^\alpha} \tag{3.11}$$

for some $p \in (p_0, 2)$*,* $\alpha > n(1/p - 1/2)$*, and* $1 \leq q \leq \infty$*. Then for any bounded Borel function F such that*

$$\sup_{t>0} \|\phi\delta_t F\|_{W_q^\alpha} < \infty \tag{3.12}$$

for some $\alpha > \max\{n(1/p - 1/2), 1/q\}$, *the operator* $F(L)$ *is bounded on* $L^r(X)$ *for all* $p < r < p'$. *In addition,*

$$\|F(L)\|_{r\to r} \le C \sup_{t>0} \|\phi\delta_t F\|_{W^\alpha_q}. \tag{3.13}$$

Proof. Observe that $\|F\|_{W^q_\alpha} \sim \|G\|_{W^q_\alpha}$ where $G(\lambda) = F(\sqrt[m]{\lambda})$. For this reason, we can replace $F(L)$ by $F(\sqrt[m]{L})$ in the proof. Let ψ be a C^∞-function, supported in $\{|\xi| \le 1/8\}$, $\int \psi(\xi)d\xi = 1$. Further set $\psi_\ell = 2^\ell\psi(2^\ell\cdot)$, $\theta_\ell = \psi_\ell - \psi_{\ell-1}$ $(\ell \ge 1)$, $\theta_0 = \psi_0$. We write

$$\begin{aligned} F &= \sum_{j\in\mathbb{Z}} \phi(2^{-j}\cdot)F \\ &= \sum_{j\in\mathbb{Z}} \sum_{\ell\ge 0} [\theta_\ell * (\phi F(2^j\cdot))](2^{-j}\cdot) \\ &= \sum_{\ell\ge 0} F_\ell \end{aligned} \tag{3.14}$$

and so for every $p < r < p'$

$$\|F(\sqrt[m]{L})\|_{r\to r} \le \sum_{\ell\ge 0} \|F_\ell(\sqrt[m]{L})\|_{r\to r}.$$

To estimate terms $\|F_\ell(\sqrt[m]{L})\|_{r\to r}$, $\ell \ge 0$ we shall apply Theorem 3.1 . Firstly, we claim that for $p < r < p'$, there exists some $\eta_r > 0$ such that

$$\sup_{t>0} \|(\phi\delta_t F_\ell)(\sqrt[m]{L})\|_{r\to r} \le C_r 2^{-\eta_r \ell} \sup_{t>0} \|\phi\delta_t F\|_{W^\alpha_q}. \tag{3.15}$$

By duality we may assume that $p < r \le 2$. Observe that $\theta_\ell * (\phi F(2^j\cdot))$ is supported in $\{\xi : \frac{1}{4} \le |\xi| \le 4\}$. If $\ell \ge 1$, we have that for $t \in [2^k, 2^{k+1}]$,

$$(\phi\delta_t F_\ell)(\sqrt[m]{L}) = \sum_{j=k-4}^{k+4} \phi(\sqrt[m]{L})[\theta_\ell * (\phi F(2^j\cdot))](2^{-j}t\sqrt[m]{L}). \tag{3.16}$$

Now we recall that if $1 \le q \le \infty$ and $\alpha - 1/q > 0$, then

$$W^\alpha_q \subseteq B^\alpha_{q,\infty} \subseteq B^{\alpha-\frac{1}{q}}_{\infty,\infty} \subseteq \Lambda_{\min\{\alpha-\frac{1}{q},\frac{1}{2}\}}$$

and $\|F\|_{\Lambda_{\min\{\alpha-\frac{1}{q},\frac{1}{2}\}}} \le C\|F\|_{W^\alpha_q}$. See, e.g., [3, Chap. VI] for more details. Hence

$$\|\phi\delta_t F\|_{\Lambda_{\min\{\alpha-1/q,1/2\}}} \le C\|\phi\delta_t F\|_{W^\alpha_q}.$$

This implies that

$$\|\theta_\ell * (\phi F(2^j\cdot))\|_\infty \le C2^{-\ell\epsilon} \sup_{t>0} \|\phi\delta_t F\|_{W^\alpha_q}$$

with $\epsilon = \min\{\alpha - 1/q,\ 1/2\}$. Hence

$$\|[\theta_\ell * (\phi F(2^j\cdot))](2^{-j}t\sqrt[m]{L})\|_{2\to 2} \le C2^{-\ell\epsilon} \sup_{t>0} \|\phi\delta_t F\|_{W^\alpha_q}.$$

By (3.16) and the fact that $\|\phi(\sqrt[m]{L})\|_{2\to 2} \le C$

$$\|(\phi\delta_t F_\ell)(\sqrt[m]{L})\|_{2\to 2} \le C2^{-\ell\epsilon} \sup_{t>0} \|\phi\delta_t F\|_{W^\alpha_q}. \tag{3.17}$$

Note that for each ℓ, the function $\theta_\ell * (\phi F(2^j\cdot))$ is supported in $\{\xi : \frac14 \le |\xi| \le 4\}$. By (3.11)

$$\begin{aligned}\|[\theta_\ell * (\phi F(2^j\cdot))](2^{-j}t\sqrt[m]{L})\|_{p\to p} &\le C\|\theta_\ell * (\phi F(2^j\cdot))\|_{W_q^\alpha}\\ &\le C\sup_{t>0}\|\phi\delta_t F\|_{W_q^\alpha}.\end{aligned}$$

(a) of Lemma 2.4 shows that $\|\phi(\sqrt[m]{L})\|_{p\to p} \le C$. By (3.16)

$$\|(\phi\delta_t F_\ell)(\sqrt[m]{L})\|_{p\to p} \le C\sup_{t>0}\|\phi\delta_t F\|_{W_q^\alpha}.$$

Then it follows from the interpolation theorem that for every $r \in (p, 2)$,

$$\|(\phi\delta_t F_\ell)(\sqrt[m]{L})\|_{r\to r} \le C_r 2^{-\eta_r \ell}\sup_{t>0}\|\phi\delta_t F\|_{W_q^\alpha}$$

with $\eta_r = \epsilon(1/r - 1/p)/(1/2 - 1/p)$, and this shows (3.15).

By Theorem 3.1 for every $M > 1 + \frac{n}{2}$,

$$\|F_\ell(\sqrt[m]{L})\|_{r\to r} \le C2^{-\eta_r\ell}\sup_{t>0}\|\phi\delta_t F\|_{W_q^\alpha}\Big\{\log\Big(2 + \frac{\sup_{t>0}\|\phi\delta_t F_\ell\|_{W_2^{M+n+1}}}{2^{-\eta_r\ell}\sup_{t>0}\|\phi\delta_t F\|_{W_q^\alpha}}\Big)\Big\}^{|\frac1r-\frac12|}. \tag{3.18}$$

Let $s \ge 1$ such that $\frac1q + \frac1s = \frac32$. The Plancherel theorem and Young's inequality yields that if $\ell \ge 1$, and $t \in [2^k, 2^{k+1}]$,

$$\begin{aligned}\|\phi(\cdot)[\theta_\ell * (\phi F(2^j\cdot))](2^{-j}t\cdot)\|_{W_2^{M+n+1}} &\le \|(1+\xi^2)^{\frac{M+n+1}{2}}\widehat{\phi F(2^j\cdot)}(\xi)\widehat{\theta_\ell}(\xi)\|_2\\ &= \|\mathscr{F}^{-1}((1+\xi^2)^{\frac{\alpha}{2}}\widehat{\phi F(2^j\cdot)}(\xi)) * \mathscr{F}^{-1}((1+\xi^2)^{\frac{M+n+1-\alpha}{2}}\widehat{\theta_\ell}(\xi))\|_2\\ &\le \|\phi F(2^j\cdot)\|_{W_q^\alpha}\|\theta_\ell\|_{W_s^{M+n+1-\alpha}}\\ &\le C2^{\ell(n+M-\alpha+\frac12+\frac1q)}\sup_{t>0}\|\phi\delta_t F\|_{W_q^\alpha}.\end{aligned}$$

Hence

$$\sup_{t>0}\|\phi\delta_t F_\ell\|_{W_2^{M+n+1}} \le C2^{\ell(n+M-\alpha+\frac12+\frac1q)}\sup_{t>0}\|\phi\delta_t F\|_{W_q^\alpha}. \tag{3.19}$$

Substituting (3.19) into (3.18), we get

$$\|F_\ell(\sqrt[m]{L})\|_{r\to r} \le C2^{-\eta_r\ell}(1+\ell)^{|\frac1r-\frac12|}\sup_{t>0}\|\phi\delta_t F\|_{W_q^\alpha}.$$

Analogously, $\|F_0(\sqrt[m]{L})\|_{r\to r} \le C\sup_{t>0}\|\phi\delta_t F\|_{W_q^\alpha}$. Summing a geometrical series we obtain $\|F(\sqrt[m]{L})\|_{r\to r} \le C\sup_{t>0}\|\phi\delta_t F\|_{W_q^\alpha}$. This completes the proof. □

4. Dyadicly supported (non-singular) spectral multipliers.

In this section, we will show that restriction type conditions can be used to study spectral multipliers corresponding to functions supported in dyadic intervals. We assume that (X, d, μ) is a metric measure space satisfying the doubling property and n is the doubling dimension from condition (1.3).

4.1. **Operators with continuous spectrum.** Consider a non-negative self-adjoint operator L and numbers p and q such that $1 \le p < 2$ and $1 \le q \le \infty$. Given $R_0 \ge 0$, we say that operator L satisfies the *local Stein-Tomas restriction type condition* if: for any $R > R_0$ and for all Borel functions F such that $\operatorname{supp} F \subset [R/2, R]$,

$$(\mathrm{ST}^{q}_{p,2,m})_{R_0} \qquad \left\| F(\sqrt[m]{L}) P_{B(x,r)} \right\|_{p \to 2} \le C V(x,r)^{\frac{1}{2}-\frac{1}{p}} (Rr)^{n(\frac{1}{p}-\frac{1}{2})} \left\| \delta_R F \right\|_q$$

for all $x \in X$ and all $r \ge 1/R$.

The condition $(\mathrm{ST}^{q}_{p,2,m})_{R_0}$ is a small modification of the restriction type condition $(\mathrm{ST}^{q}_{p,2,m})$ Namely here we consider function supported in the interval $[R/2, R]$ rather then $[0, R]$, which allows us to study localized version of spectral multipliers, see Theorems 4.2 and 6.8 below.

Note that condition $(\mathrm{ST}^{q}_{p,2,m})$ implies $(\mathrm{ST}^{q}_{p,2,m})_{R_0}$ for all $R_0 \ge 0$. If in addition we assume that $\chi_{\{0\}}(\sqrt{L}) = 0$ then for $R_0 = 0$ condition $(\mathrm{ST}^{q}_{p,2,m})_{R_0}$ implies $(\mathrm{ST}^{q}_{p,2,m})$.

We say that L satisfies L^p *to* $L^{p'}$ *restriction estimates* if there exists $\lambda_0 \ge 0$ such that the spectral measure $dE_{\sqrt[m]{L}}(\lambda)$ maps $L^p(X)$ to $L^{p'}(X)$ for some $p < 2$, with an operator norm estimate

$$(\mathrm{R}_{p,m})_{\lambda_0} \qquad \left\| dE_{\sqrt[m]{L}}(\lambda) \right\|_{p \to p'} \le C \lambda^{n(\frac{1}{p}-\frac{1}{p'})-1} \quad \text{for all } \lambda \ge \lambda_0,$$

where n is as in condition (1.3) and p' is conjugate of p, i.e., $1/p + 1/p' = 1$.

Proposition 4.1. *Let $1 \le p < 2$ and $R_0 \ge 0$. Suppose that there exists a constant $C > 0$ such that $C^{-1} r^n \le V(x,r) \le C r^n$ for all $x \in X$ and $r > 0$. Then conditions $(\mathrm{R}_{p,m})_{R_0/2}$, $(\mathrm{ST}^{2}_{p,2,m})_{R_0}$ and $(\mathrm{ST}^{1}_{p,p',m})_{R_0}$ are equivalent.*

Proof. The proof is similar to that of Proposition 2.4 of [11] with minor modifications. We give a brief argument of this proof for completeness and readers' convenience.

We first show the implication $(\mathrm{R}_{p,m})_{R_0/2} \Rightarrow (\mathrm{ST}^{1}_{p,p',m})_{R_0}$. Suppose that F is a Borel function such that $\operatorname{supp} F \subset [R/2, R]$ for some $R > R_0$. Then by $(\mathrm{R}_{p,m})_{R_0/2}$

$$\begin{aligned}
\left\| F(\sqrt[m]{L}) P_{B(x,r)} \right\|_{p \to p'} &\le \int_0^\infty |F(\lambda)| \| dE_{\sqrt[m]{L}}(\lambda) \|_{p \to p'} d\lambda \\
&\le C \int_{R/2}^{R} |F(\lambda)| \lambda^{n(\frac{1}{p}-\frac{1}{p'})-1} d\lambda \\
&\le C R^{n(\frac{1}{p}-\frac{1}{p'})-1} \int_{R/2}^{R} |F(\lambda)| d\lambda \\
&\le C V(x,r)^{\frac{1}{p'}-\frac{1}{p}} (rR)^{n(\frac{1}{p}-\frac{1}{p'})} \| \delta_R F \|_1,
\end{aligned}$$

where in the last inequality we used the assumption that $V(x,r) \le C r^n$.

Next we prove that $(\mathrm{ST}^{1}_{p,p',m})_{R_0} \Rightarrow (\mathrm{ST}^{2}_{p,2,m})_{R_0}$. Note that $V(x,r) \sim r^n$ for every $x \in X$ and $r > 0$. Letting $r \to \infty$ we obtain from $(\mathrm{ST}^{1}_{p,p',m})_{R_0}$

$$\left\| F(\sqrt[m]{L}) \right\|_{p \to p'} \le C R^{n(\frac{1}{p}-\frac{1}{p'})} \| \delta_R F \|_1, \quad R > R_0.$$

By T^*T argument

$$\left\| F(\sqrt[m]{L}) \right\|_{p \to 2}^2 = \left\| |F|^2(\sqrt[m]{L}) \right\|_{p \to p'} \le C R^{2n(\frac{1}{p}-\frac{1}{2})} \| \delta_R F \|_2^2.$$

Hence

$$\left\| F(\sqrt[m]{L}) P_{B(x,r)} \right\|_{p \to 2} \le \left\| F(\sqrt[m]{L}) \right\|_{p \to 2} \le C V(x,r)^{\frac{1}{2}-\frac{1}{p}} (Rr)^{n(\frac{1}{p}-\frac{1}{2})} \| \delta_R F \|_2.$$

This gives $(\mathrm{ST}^2_{\mathrm{p,2,m}})_{\mathrm{R_0}}$.

Now, we prove the remaining implication $(\mathrm{ST}^2_{\mathrm{p,2,m}})_{\mathrm{R_0}} \Rightarrow (\mathrm{R_{p,m}})_{\mathrm{R_0/2}}$. By volume estimate $V(x,r) \geq C^{-1}r^n$ and condition $(\mathrm{ST}^2_{\mathrm{p,2,m}})_{\mathrm{R_0}}$

$$\big\|F(\sqrt[m]{L})P_{B(x,r)}\big\|_{p\to 2} \leq CR^{n(\frac{1}{p}-\frac{1}{2})}\big\|\delta_R F\big\|_2 \tag{4.1}$$

for all Borel functions F such that $\operatorname{supp} F \subset [R/2, R]$, all $R > R_0$, all $x \in X$ and $r \geq 1/R$. Taking the limit $r \to \infty$ gives

$$\big\|F(\sqrt[m]{L})\big\|_{p\to 2} \leq CR^{n(\frac{1}{p}-\frac{1}{2})}\big\|\delta_R F\big\|_2. \tag{4.2}$$

Let $\epsilon < R/4$. Putting $F = \chi_{(\lambda-\varepsilon,\lambda+\varepsilon]}$ and $R = \lambda+\epsilon$ in (4.2) yields

$$\begin{aligned}\Big\|\varepsilon^{-1}E_{\sqrt[m]{L}}(\lambda-\varepsilon,\lambda+\varepsilon]\Big\|_{p\to p'} &= \varepsilon^{-1}\Big\|E_{\sqrt[m]{L}}(\lambda-\varepsilon,\lambda+\varepsilon]\Big\|^2_{p\to 2}\\ &\leq C\varepsilon^{-1}(\lambda+\varepsilon)^{2n(\frac{1}{p}-\frac{1}{2})}\Big\|\chi_{(\frac{\lambda-\varepsilon}{\lambda+\epsilon},1]}\Big\|_2^2\\ &\leq C(\lambda+\epsilon)^{n(\frac{1}{p}-\frac{1}{p'})-1}.\end{aligned}$$

Taking $\varepsilon \to 0$ yields condition $(\mathrm{R_{p,m}})_{\mathrm{R_0/2}}$ (see Proposition 1, Chapter XI, [52]). □

The following result and its proof are inspired by Theorem 1.1 of [28]. See also Theorem 3.1 of [11].

Theorem 4.2. *Suppose that (X,d,μ) and a non-negative self-adjoint operator L acting on $L^2(X)$ satisfies estimates $(\mathrm{DG_m})$ and $(\mathrm{G_{p,2,m}})$ for some $1 \leq p < 2$. Next assume that condition $(\mathrm{ST}^q_{\mathrm{p,2,m}})_{\mathrm{R_0}}$ holds for some $R_0 \geq 0$ and and $1 \leq q \leq \infty$ and that F is a bounded Borel function such that $\operatorname{supp} F \subseteq [1/4,4]$ and $F \in W^\alpha_q(\mathbb{R})$ for some $\alpha > n(1/p-1/2)$.*

Then for every $p < r \leq 2$, $F(t\sqrt[m]{L})$ is bounded on $L^r(X)$,

$$\sup_{t<1/(8R_0)} \|F(t\sqrt[m]{L})\|_{r\to r} \leq C_r\|F\|_{W^\alpha_q} \tag{4.3}$$

and

$$\sup_{t\geq 1/(8R_0)} \|F(t\sqrt[m]{L})\|_{r\to r} \leq C_r\|F\|_{W^\alpha_\infty}. \tag{4.4}$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R})$ be a function such that $\operatorname{supp}\phi \subseteq \{\xi : 1/4 \leq |\xi| \leq 1\}$ and $\sum_{\ell\in\mathbb{Z}}\phi(2^{-\ell}\lambda) = 1$ for all $\lambda > 0$. Set $\phi_0(\lambda) = 1 - \sum_{\ell=0}^\infty \phi(2^{-\ell}\lambda)$,

$$G^{(0)}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_0(\tau)\hat{G}(\tau)e^{i\tau\lambda}\,d\tau \tag{4.5}$$

and

$$G^{(\ell)}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(2^{-\ell}\tau)\hat{G}(\tau)e^{i\tau\lambda}\,d\tau, \tag{4.6}$$

where $G(\lambda) = F(\sqrt[m]{\lambda})e^\lambda$. Note that in by the Fourier inversion formula

$$G(\lambda) = \sum_{\ell=0}^\infty G^{(\ell)}(\lambda).$$

Then

$$F(\sqrt[m]{\lambda}) = G(\lambda)e^{-\lambda} \;=\; \sum_{\ell=0}^{\infty} G^{(\ell)}(\lambda)e^{-\lambda} =: \sum_{\ell=0}^{\infty} F^{(\ell)}(\sqrt[m]{\lambda}) \tag{4.7}$$

so

$$\left\|F(t\sqrt[m]{L})\right\|_{r\to r} \;\le\; \sum_{\ell=0}^{\infty} \left\|F^{(\ell)}(t\sqrt[m]{L})\right\|_{r\to r}, \quad r\in(p,2). \tag{4.8}$$

Next we fix $\varepsilon > 0$ such that $2n\varepsilon(1/p-1/2) \le \alpha - n(1/p-1/2)$. For every $t>0$ and every ℓ set $\rho_\ell = 2^{\ell(1+\varepsilon)}t$. Then we choose a sequence $(x_n)\in X$ such that $d(x_i,x_j) > \rho_\ell/10$ for $i\ne j$ and $\sup_{x\in X}\inf_i d(x,x_i) \le \rho_\ell/10$. Such sequence exists because X is separable. Now set $B_i = B(x_i,\rho_\ell)$ and define $\widetilde{B}_i$ by the formula

$$\widetilde{B}_i = \bar{B}\Big(x_i, \frac{\rho_\ell}{10}\Big) \setminus \bigcup_{j<i} \bar{B}\Big(x_j, \frac{\rho_\ell}{10}\Big),$$

where $\bar{B}(x,\rho_\ell) = \{y\in X\colon d(x,y)\le \rho_\ell\}$. Note that for $i\ne j$, $B(x_i,\frac{\rho_\ell}{20}) \cap B(x_j,\frac{\rho_\ell}{20}) = \emptyset$.

Observe that for every $k\in\mathbb{N}$,

$$\begin{aligned}\sup_i \#\{j:\ d(x_i,x_j)\le 2^k\rho_\ell\} \;&\le\; \sup_{d(x,y)\le 2^k\rho_\ell} \frac{V(x,2^{k+1}\rho_\ell)}{V(y,\frac{\rho_\ell}{20})}\\ &\le\; C2^{kn}\sup_y \frac{V(y,2^{k+2}\rho_\ell)}{V(y,\frac{\rho_\ell}{20})} \le C2^{kn}.\end{aligned} \tag{4.9}$$

Set $\mathcal{D}_{\rho_\ell} = \{(x,\,y)\in X\times X : d(x,\,y)\le\rho_\ell\}$. It is not difficult to see that

$$\mathcal{D}_{\rho_\ell} \subseteq \bigcup_{\{i,j\colon d(x_i,x_j)<2\rho_\ell\}} \widetilde{B}_i\times\widetilde{B}_j \subseteq \mathcal{D}_{4\rho_\ell}. \tag{4.10}$$

Now let $\psi\in C_c^\infty(1/16,16)$ be a such function that $\psi(\lambda)=1$ for $\lambda\in(1/8,8)$, and we decompose

$$\begin{aligned}F^{(\ell)}(t\sqrt[m]{L})f \;&=\; \sum_{i,j\colon d(x_i,x_j)<2\rho_\ell} P_{\widetilde{B}_i}[\psi F^{(\ell)}(t\sqrt[m]{L})]P_{\widetilde{B}_j} f\\ &\quad+ \sum_{i,j\colon d(x_i,x_j)<2\rho_\ell} P_{\widetilde{B}_i}[(1-\psi)F^{(\ell)}(t\sqrt[m]{L})]P_{\widetilde{B}_j} f\\ &\quad+ \sum_{i,j\colon d(x_i,x_j)\ge 2\rho_\ell} P_{\widetilde{B}_i}F^{(\ell)}(t\sqrt[m]{L})P_{\widetilde{B}_j} f = I + I\!I + I\!I\!I.\end{aligned} \tag{4.11}$$

Estimate for I . By Hölder's inequality,

$$\begin{aligned}\|\sum_{i,j\colon d(x_i,x_j)<2\rho_\ell} P_{\widetilde{B}_i}(\psi F^{(\ell)})(t\sqrt[m]{L})P_{\widetilde{B}_j} f\|_r^r &= \sum_i \|\sum_{j\colon d(x_i,x_j)<2\rho_\ell} P_{\widetilde{B}_i}(\psi F^{(\ell)})(t\sqrt[m]{L})P_{\widetilde{B}_j} f\|_r^r\\ &\le C\sum_i \sum_{j\colon d(x_i,x_j)<2\rho_\ell} \|P_{\widetilde{B}_i}(\psi F^{(\ell)})(t\sqrt[m]{L})P_{\widetilde{B}_j} f\|_r^r\\ \le C\sum_i \sum_{j\colon d(x_i,x_j)<2\rho_\ell} &V(\widetilde{B}_i)^{r(\frac{1}{r}-\frac{1}{2})}\|P_{\widetilde{B}_i}(\psi F^{(\ell)})(t\sqrt[m]{L})P_{\widetilde{B}_j} f\|_2^r\\ &\le C\sum_j V(B_j)^{r(\frac{1}{r}-\frac{1}{2})}\|(\psi F^{(\ell)})(t\sqrt[m]{L})P_{\widetilde{B}_j} f\|_2^r\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_j V(B_j)^{r(\frac{1}{r}-\frac{1}{2})}\|(\psi F^{(\ell)})(t\sqrt[m]{L})P_{\widetilde{B}_j}\|_{p\to 2}^r\|P_{\widetilde{B}_j}\|_{r\to p}^r\|P_{\widetilde{B}_j}f\|_r^r\\
&\le C\sup_{x\in X}\{V(x,\rho_\ell)^{r(\frac{1}{p}-\frac{1}{2})}\|(\psi F^{(\ell)})(t\sqrt[m]{L})P_{B(x,\rho_\ell)}\|_{p\to 2}^r\}\sum_j\|P_{\widetilde{B}_j}f\|_r^r\\
&= C\sup_{x\in X}\{V(x,\rho_\ell)^{r(\frac{1}{p}-\frac{1}{2})}\|(\psi F^{(\ell)})(t\sqrt[m]{L})P_{B(x,\rho_\ell)}\|_{p\to 2}^r\}\|f\|_r^r.
\end{aligned}
\tag{4.12}
$$

Case 1. $t<1/(8R_0)$.

We assume that $\psi\in C_c(1/16,16)$ so we can write $(\psi F^{(\ell)})(t\sqrt[m]{L})=\sum_{k=0}^7(\chi_{[2^{k-4},2^{k-3})}\psi F^{(\ell)})(t\sqrt[m]{L})$. If $t<1/(8R_0)$, then we use condition $(\mathrm{ST^q_{p,2,m}})_{\mathrm{R_0}}$ to show that for every $\ell\ge 4$,

$$
\begin{aligned}
\left\|(\psi F^{(\ell)})(t\sqrt[m]{L})P_{B(x,\rho_\ell)}\right\|_{p\to 2} &\le CV(x,\rho_\ell)^{\frac12-\frac1p}2^{\ell(1+\varepsilon)n(\frac1p-\frac12)}\sum_{k=0}^7\left\|\delta_{2^{k-3}t^{-1}}(\psi F^{(\ell)})(t\cdot)\right\|_q\\
&\le CV(x,\rho_\ell)^{\frac12-\frac1p}2^{\ell(1+\varepsilon)n(\frac1p-\frac12)}\|G^{(\ell)}\|_q.
\end{aligned}
\tag{4.13}
$$

Note that by Proposition 2.3, $(\mathrm{G_{p,2,m}})$ implies $(\mathrm{ST^\infty_{p,2,m}})$. From this, it can be verified that for $\ell=0,1,2,3$, $\left\|(\psi F^{(\ell)})(t\sqrt[m]{L})P_{B(x,\rho_\ell)}\right\|_{p\to 2}\le CV(x,\rho_\ell)^{\frac12-\frac1p}\|F\|_q$. Hence

$$
\begin{aligned}
\sum_{\ell=0}^\infty\sup_{x\in X}\{V(x,\rho_\ell)^{\frac1p-\frac12}\left\|(\psi F^{(\ell)})(t\sqrt[m]{L})P_{B(x,\rho_\ell)}\right\|_{p\to 2}\} &\le C\|F\|_q+C\sum_{\ell=4}^\infty 2^{\ell(1+\varepsilon)n(\frac1p-\frac12)}\|G^{(\ell)}\|_q\\
&\le C\|F\|_q+C\|G\|_{B_{q,1}^{n(\frac1p-\frac12)+\delta}},
\end{aligned}
\tag{4.14}
$$

where $\delta=\varepsilon n(\frac1p-\frac12)$ and the last equality follows from definition of Besov space. See, e.g., [3, Chap. VI]. Since $2\delta<\alpha-n(\frac1p-\frac12)$, we have that $W_q^\alpha\subseteq B_{q,1}^{n(1/p-1/2)+\delta}$ with $\|G\|_{B_{q,1}^{n(1/p-1/2)+\delta}}\le C_\alpha\|G\|_{W_q^\alpha}$, see again [3]. However, $\operatorname{supp}F\subseteq[1/4,4]$ so $\|G\|_{W_q^\alpha}\le\|F\|_{W_q^\alpha}$. Hence the forgoing estimates give

$$
\sum_{\ell=0}^\infty\sup_{x\in X}\{V(x,\rho_\ell)^{\frac1p-\frac12}\left\|(\psi F^{(\ell)})(t\sqrt[m]{L})P_{B(x,\rho_\ell)}\right\|_{p\to 2}\}\le C\|F\|_{W_q^\alpha}.
\tag{4.15}
$$

Case 2. $t\ge 1/(8R_0)$.

Note that by Proposition 2.3, $(\mathrm{G_{p,2,m}})$ implies $(\mathrm{ST^\infty_{p,2,m}})$. At the step (4.13) we use the condition $(\mathrm{ST^\infty_{p,2,m}})$ in place of $(\mathrm{ST^q_{p,2,m}})_{\mathrm{R_0}}$, and the similar argument as above shows

$$
\sum_{\ell=0}^\infty\sup_{x\in X}\{V(x,\rho_\ell)^{\frac1p-\frac12}\left\|(\psi F^{(\ell)})(t\sqrt[m]{L})P_{B(x,\rho_\ell)}\right\|_{p\to 2}\}\le C\|F\|_{W_\infty^\alpha}.
\tag{4.16}
$$

.

Estimate of $I\!I$. Repeat an argument leading up to (4.12), it is easy to see that

$$
\begin{aligned}
\Big\|\sum_{i,j:\,d(x_i,x_j)<2\rho_\ell}P_{\widetilde{B}_i}((1-\psi)F^{(\ell)})(t\sqrt[m]{L})P_{\widetilde{B}_j}f\Big\|_r &\le C\|((1-\psi)F^{(\ell)})(t\sqrt[m]{L})P_{B(x,\rho_\ell)}\|_{r\to r}\|f\|_r\\
&\le C\|((1-\psi)F^{(\ell)})(t\sqrt[m]{L})\|_{r\to r}\|f\|_r,
\end{aligned}
$$

where, for a fixed N, one has the uniform estimates

$$
\left|\Big(\frac{d}{d\lambda}\Big)^\kappa((1-\psi)F^{(\ell)})(\lambda)\right|\le C_\kappa 2^{-\ell N}(1+|\lambda|)^{-N}\|F\|_{W_q^\alpha}.
$$

But (a) of Lemma 2.4 then implies that for every $r \in (p, 2)$,

$$\|((1-\psi)F^{(\ell)})(t\sqrt[m]{L})\|_{r\to r} \le C2^{-\ell N}\|F\|_{W^\alpha_q}.$$

Therefore,

$$\sum_{\ell=0}^{\infty} \|((1-\psi)F^{(\ell)})(t\sqrt[m]{L})\|_{r\to r} \le C\|F\|_{W^\alpha_q}. \tag{4.17}$$

Estimate of $I\!I\!I$. Note that

$$\begin{aligned}\Big\|\sum_{i,j:\, d(x_i,x_j)\ge 2^{\ell(1+\varepsilon)}t} P_{\widetilde{B}_i}F^{(\ell)}(t\sqrt[m]{L})P_{\widetilde{B}_j}f\Big\|_r^r &= \sum_i \Big\|\sum_{j:\, d(x_i,x_j)\ge 2^{\ell(1+\varepsilon)}t} P_{\widetilde{B}_i}F^{(\ell)}(t\sqrt[m]{L})P_{\widetilde{B}_j}f\Big\|_r^r \\ &\le \sum_i\Big(\sum_{j:\, d(x_i,x_j)\ge 2^{\ell(1+\varepsilon)}t} \big\|P_{\widetilde{B}_i}F^{(\ell)}(t\sqrt[m]{L})P_{\widetilde{B}_j}f\big\|_r\Big)^r.\end{aligned}$$

To go further, we need the following lemma.

Lemma 4.3. *Suppose that assumptions of Theorem 4.2 are fulfilled. Let $r \in (p, 2)$. For all $\ell = 0, 1, 2, \dots$ and all x_i, x_j with $d(x_i, x_j) \ge 2^{\ell(1+\varepsilon)}t$, there exist some positive constants $C, c_1, c_2 > 0$ such that*

$$\big\|P_{\widetilde{B}_i}F^{(\ell)}(t\sqrt[m]{L})P_{\widetilde{B}_j}f\big\|_r \le Ce^{-c_1 2^{\frac{\varepsilon\ell m}{m-1}}}\exp\Big(-c_2\Big(\frac{d(x_i,x_j)}{2^\ell t}\Big)^{\frac{m}{m-1}}\Big)\|F\|_q\|P_{\widetilde{B}_j}f\|_r.$$

Proof. By the formula (4.6),

$$\begin{aligned}&\big\|P_{\widetilde{B}_i}F^{(\ell)}(t\sqrt[m]{L})P_{\widetilde{B}_j}f\big\|_r \\ &\le \ C\|P_{\widetilde{B}_j}f\|_r\int_{-\infty}^{+\infty}|\phi(2^{-\ell}\tau)\hat{G}(\tau)|\big\|P_{\widetilde{B}_i}e^{(i\tau-1)t^m L}P_{\widetilde{B}_j}\big\|_{r\to r}\,d\tau,\end{aligned} \tag{4.18}$$

where $G(\lambda) = F(\sqrt[m]{\lambda})e^{\lambda}$. Recall that hypothesis $(\mathrm{DG_m})$ and $(\mathrm{G_{p,2,m}})$ imply $(\mathrm{GGE_{r,2}})$. This, in combination with Lemma 2.1 (with $z = (i\tau-1)t^m$), gives

$$\begin{aligned}&\big\|P_{\bar{B}(x_i,\frac{\rho_\ell}{10})}e^{(i\tau-1)t^m L}P_{\bar{B}(x_j,\frac{\rho_\ell}{10})}\big\|_{r\to 2} \\ &\le \ CV(x_i,\frac{\rho_\ell}{10})^{-(\frac1r-\frac12)}\Big(1+\frac{\rho_\ell}{10t\sqrt{\tau^2+1}}\Big)^{n(\frac1r-\frac12)}\Big(\sqrt{1+\tau^2}\Big)^{n(\frac1r-\frac12)}\exp\Big(-c\Big(\frac{d(x_i,x_j)}{t\sqrt{\tau^2+1}}\Big)^{\frac{m}{m-1}}\Big),\end{aligned}$$

which shows

$$\begin{aligned}&\big\|P_{\bar{B}(x_i,\frac{\rho_\ell}{10})}e^{(i\tau-1)t^m L}P_{\bar{B}(x_j,\frac{\rho_\ell}{10})}\big\|_{r\to r} \\ &\le \ \big\|P_{\bar{B}(x_i,\frac{\rho_\ell}{10})}e^{(i\tau-1)t^m L}P_{\bar{B}(x_j,\frac{\rho_\ell}{10})}\big\|_{r\to 2}\big\|P_{\bar{B}(x_i,\frac{\rho_\ell}{10})}\big\|_{2\to r} \\ &\le \ C\Big(1+\frac{\rho_\ell}{10t\sqrt{\tau^2+1}}\Big)^{n(\frac1r-\frac12)}\Big(\sqrt{1+\tau^2}\Big)^{n(\frac1r-\frac12)}\exp\Big(-c\Big(\frac{d(x_i,x_j)}{t\sqrt{\tau^2+1}}\Big)^{\frac{m}{m-1}}\Big).\end{aligned}$$

Hence, if $\tau \in [2^{\ell-2}, 2^\ell]$, then

$$\begin{aligned}\big\|P_{\widetilde{B}_i}e^{(i\tau-1)t^m L}P_{\widetilde{B}_j}\big\|_{r\to r} &\le \big\|P_{\bar{B}(x_i,\frac{\rho_\ell}{10})}e^{(i\tau-1)t^m L}P_{\bar{B}(x_j,\frac{\rho_\ell}{10})}\big\|_{r\to r} \\ &\le C2^{\ell n(1+\varepsilon)(\frac1r-\frac12)}\exp\Big(-c\Big(\frac{d(x_i,x_j)}{2^\ell t}\Big)^{\frac{m}{m-1}}\Big).\end{aligned}$$

Substituting the above inequality into (4.18) and using the fact that $\|\hat{G}\|_\infty \leq \|F\|_q$ yield that for $d(x_i, x_j) \geq 2^{\ell(1+\varepsilon)}t$,

$$\begin{aligned}\big\|P_{\widetilde{B}_i} F^{(\ell)}(t\sqrt[m]{L})P_{\widetilde{B}_j} f\big\|_r &\leq C2^{\ell n(1+\varepsilon)(\frac{1}{r}-\frac{1}{2})+1} e^{-c_2 2^{\frac{\varepsilon\ell m}{m-1}}} \exp\Big(-c_2\Big(\frac{d(x_i,x_j)}{2^\ell t}\Big)^{\frac{m}{m-1}}\Big)\|F\|_q\|P_{\widetilde{B}_j} f\|_r\\ &\leq Ce^{-c_1 2^{\frac{\varepsilon\ell m}{m-1}}} \exp\Big(-c_2\Big(\frac{d(x_i,x_j)}{2^\ell t}\Big)^{\frac{m}{m-1}}\Big)\|F\|_q\|P_{\widetilde{B}_j} f\|_r\end{aligned}$$

with $c_1 = c/4$ and $c_2 = c/2$. This proves Lemma 4.3. □

Back to the proof of Theorem 4.2. By (4.9) for every i

$$\begin{aligned}\sum_{j:\, d(x_i,x_j)\geq 2^{\ell(1+\varepsilon)}t} \exp\Big(-c_2\Big(\frac{d(x_i,x_j)}{2^\ell t}\Big)^{\frac{m}{m-1}}\Big) &\leq \sum_{k=1}^{\infty} \sum_{j:\, 2^k 2^{\ell(1+\varepsilon)}t\leq d(x_i,x_j)<2^{k+1}2^{\ell(1+\varepsilon)}t} \exp\big(-c_2 2^{\frac{m(k+\ell\varepsilon)}{m-1}}\big)\\ &\leq \sum_{k=1}^{\infty} 2^{2kn}\exp\big(-c_2 2^{\frac{m(k+\ell\varepsilon)}{m-1}}\big) \leq C,\end{aligned}$$

which, together with Lemma 4.3 and the Cauchy-Schwarz inequality, yields

$$\begin{aligned}&\Big\|\sum_{i,j:\, d(x_i,x_j)\geq 2^{\ell(1+\varepsilon)}t} P_{\widetilde{B}_i} F^{(\ell)}(t\sqrt[m]{L})P_{\widetilde{B}_j} f\Big\|_r^r\\ &\quad\leq Ce^{-c_1 r 2^{\frac{\varepsilon\ell m}{m-1}}}\|F\|_q^r \sum_i \Big\{\sum_{j:\, d(x_i,x_j)\geq 2^{\ell(1+\varepsilon)}t} \exp\Big(-c_2\Big(\frac{d(x_i,x_j)}{2^\ell t}\Big)^{\frac{m}{m-1}}\Big)\|P_{\widetilde{B}_j} f\|_r\Big\}^r\\ &\quad\leq Ce^{-c_1 r 2^{\frac{\varepsilon\ell m}{m-1}}}\|F\|_q^r \sum_j \|P_{\widetilde{B}_j} f\|_r^r \sum_{i:\, d(x_i,x_j)\geq 2^{\ell(1+\varepsilon)}t} \exp\Big(-c_2\Big(\frac{d(x_i,x_j)}{2^\ell t}\Big)^{\frac{m}{m-1}}\Big)\\ &\quad\leq Ce^{-c_1 r 2^{\frac{\varepsilon\ell m}{m-1}}}\|F\|_q^r\|f\|_r^r.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\ell=0}^{\infty}\Big\|\sum_{i,j:\, d(x_i,x_j)\geq 2^{\ell(1+\varepsilon)}t} P_{\widetilde{B}_i} F^{(\ell)}(t\sqrt[m]{L})P_{\widetilde{B}_j} f\Big\|_r &\leq C\sum_{\ell=0}^{\infty} e^{-c_1 2^{\frac{\varepsilon\ell m}{m-1}}}\|F\|_q\|f\|_r\\ &\leq C\|F\|_q\|f\|_r. \qquad (4.19)\end{aligned}$$

Estimates (4.3) and (4.4) then follow from (4.8), (4.11), (4.12), (4.15), (4.16), (4.17) and (4.19). This completes the proof of Theorem 4.2. □

As we explained in the introduction, a standard application of spectral multiplier theorems is Bochner-Riesz means. Such application is also a good test to check if the considered multiplier result is sharp. Let us recall that Bochner-Riesz means of order δ for a non-negative self-adjoint operator L are defined by the formula

$$S_R^\delta(L) = \Big(I - \frac{L}{R^m}\Big)_+^\delta, \quad R > 0. \qquad (4.20)$$

The case $\delta = 0$ corresponds to the spectral projector $E_{\sqrt[m]{L}}[0, R]$. For $\delta > 0$ we think of (4.20) as a smoothed version of this spectral projector; the larger δ, the more smoothing. Bochner-Riesz summability on L^p describes the range of δ for which $S_R^\delta(L)$ are bounded on L^p, uniformly in R.

Corollary 4.4. *Suppose that the operator L satisfies Davies-Gaffney estimates* ($\mathrm{DG_m}$) *and condition* ($\mathrm{ST^q_{p,2,m}}$) *with some $1 \le p < 2$ and $1 \le q \le \infty$.*

Then

$$\sup_{R>0} \Big\| \Big(I - \frac{L}{R^m}\Big)_+^{\delta} \Big\|_{r\to r} \le C \tag{4.21}$$

for all $p < r \le 2$ and $\delta > n(1/r - 1/2) - 1/q$.

Proof. Let $F(\lambda) = (1-\lambda^m)_+^\delta$. We set

$$F(\lambda) = F(\lambda)\phi(\lambda^m) + F(\lambda)(1-\phi(\lambda^m)) =: F_1(\lambda^m) + F_2(\lambda^m),$$

where $\phi \in C^\infty(\mathbb{R})$ is supported in $\{\xi : |\xi| \ge 1/4\}$ and $\phi = 1$ for all $|\xi| \ge 1/2$. Observe that $F \in W^\alpha_q$ if and only if $\delta > \alpha - 1/q$. This, in combination with Theorem 4.2, shows that for all $p < r \le 2$ and $\delta > n(1/r - 1/2) - 1/q$, $\sup_{R>0} \|F_1(L/R^m)\|_{r\to r} \le C$. On the other hand, it follows from Lemma 2.4 that $\|F_2(L/R^m)\|_{r\to r} \le C$ uniformly in $R > 0$. This completes the proof of estimate (4.21). □

4.2. **Operators with non-empty point spectrum.** It is not difficult to see that condition ($\mathrm{ST^q_{p,2,m}}$) with some $q < \infty$ implies that the set of point spectrum of L is empty. Indeed, one has for all $0 \le a < R$ and $x \in X$,

$$\Big\| \mathbb{1}_{\{a\}}(\sqrt[m]{L}) P_{B(x,r)} \Big\|_{p\to 2} \le C V(x,r)^{\frac12 - \frac1p} (rR)^{n(\frac1p - \frac12)} \Big\| \mathbb{1}_{\{a\}}(R\cdot) \Big\|_q = 0, \quad Rr \ge 1$$

and therefore $\mathbb{1}_{\{a\}}(\sqrt[m]{L}) = 0$ so the point spectrum of L is empty, see [22]. In particular, ($\mathrm{ST^q_{p,2,m}}$) cannot hold for any $q < \infty$ for elliptic operators on compact manifolds or for the harmonic oscillator. To be able to study these operators as well, we introduce a variation of condition ($\mathrm{ST^q_{p,2,m}}$). Following [17, 22], for an even Borel function F with $\operatorname{supp} F \subseteq [-1, 1]$ we define the norm $\|F\|_{N,q}$ by

$$\|F\|_{N,q} = \left(\frac{1}{2N} \sum_{\ell=1-N}^{N} \sup_{\lambda \in [\frac{\ell-1}{N}, \frac{\ell}{N})} |F(\lambda)|^q \right)^{1/q},$$

where $q \in [1, \infty)$ and $N \in \mathbb{N}$. For $q = \infty$, we put $\|F\|_{N,\infty} = \|F\|_\infty$. It is obvious that $\|F\|_{N,q}$ increases monotonically in q.

Consider a non-negative self-adjoint operator L and numbers p and q such that $1 \le p < 2$ and $1 \le q \le \infty$. Following [11], we shall say that L satisfies the *Sogge spectral cluster condition* if: for a fixed natural number κ and for all $N \in \mathbb{N}$ and all even Borel functions F such that $\operatorname{supp} F \subseteq [-N, N]$,

$$\Big\| F(\sqrt[m]{L}) P_{B(x,r)} \Big\|_{p\to 2} \le C V(x,r)^{\frac12 - \frac1p} (Nr)^{n(\frac1p - \frac12)} \|\delta_N F\|_{N^\kappa, q} \tag{$\mathrm{SC^{q,\kappa}_{p,2,m}}$}$$

for all $x \in X$ and $r \ge 1/N$. For $q = \infty$, ($\mathrm{SC^{\infty,\kappa}_{p,2,m}}$) is independent of κ so we write it as ($\mathrm{SC^{\infty}_{p,2,m}}$).

Remark 4.5. *It is easy to check that for $\kappa \ge 1$,* ($\mathrm{SC^{q,\kappa}_{p,2,m}}$) *implies* ($\mathrm{SC^{q,1}_{p,2,m}}$)*. Moreover, if $\mu(X) < \infty$, then conditions* ($\mathrm{SC^{\infty}_{p,2,m}}$) *and* ($\mathrm{ST^{\infty}_{p,2,m}}$) *are equivalent(see Proposition 3.11,* [11]*).*

The next theorem is a version of Theorem 4.2 suitable for the operators satisfying condition ($\mathrm{SC^{q,\kappa}_{p,2,m}}$).

Theorem 4.6. *Suppose the operator L satisfies Davies-Gaffney estimates* $(\mathrm{DG_m})$, *conditions* $(\mathrm{G_{p,2,m}})$ *and condition* $(\mathrm{SC^{q,\kappa}_{p,2,m}})$ *for a fixed $\kappa \in \mathbb{N}$ and some p, q such that $1 \le p < 2$ and $1 \le q \le \infty$. In addition, we assume that for any $\varepsilon > 0$ there exists a constant C_ε such that for all $N \in \mathbb{N}$ and all even Borel functions F such that supp $F \subset [-N, N]$,*

$$(\mathrm{AB_{p,m}}) \qquad \|F(\sqrt[m]{L})\|_{p\to p} \le C_\varepsilon N^{\kappa n(\frac{1}{p}-\frac{1}{2})+\varepsilon}\|\delta_N F\|_{N^\kappa,q}.$$

Let $p < r \le 2$. Then for any function F such that $\operatorname{supp} F \subseteq [1/4, 4]$ *and $\|F\|_{W^\alpha_q} < \infty$ for some $\alpha > \max\{n(1/p - 1/2), 1/q\}$, the operator $F(t\sqrt[m]{L})$ is bounded on $L^p(X)$ for all $t > 0$. In addition,*

$$\sup_{t>0} \|F(t\sqrt[m]{L})\|_{r\to r} \le C\|F\|_{W^\alpha_q}. \tag{4.22}$$

Note that condition $(\mathrm{SC^{q,\kappa}_{p,2,m}})$ is weaker than $(\mathrm{ST^{q}_{p,2,m}})$ and we need a priori estimate $(\mathrm{AB_{p,m}})$ in Theorem 4.6. See also [17, Theorem 3.6] and [22, Theorem 3.2] for related results. Once $(\mathrm{SC^{q,\kappa}_{p,2,m}})$ is proved, a priori estimate $(\mathrm{AB_{p,m}})$ is not difficult to check in general.

Proposition 4.7. *Suppose that $\mu(X) < \infty$ and $(\mathrm{SC^{q,1}_{p,2,m}})$ for some p, q such that $1 \le p < 2$ and $1 \le q \le \infty$. Then*

$$\|F(\sqrt[m]{L})\|_{p\to p} \le CN^{n(\frac{1}{p}-\frac{1}{2})}\|\delta_N F\|_{N,q}$$

for all $N \in \mathbb{N}$ and all Borel functions F such that supp $F \subseteq [-N, N]$.

Proof. We follow Proposition 3.7 of [22] to prove the result (see also [22]). Since $\mu(X) < \infty$, we may assume that $X = B(x_0, 1)$ for some $x_0 \in X$. It follows from Hölder's inequality and condition $(\mathrm{SC^{q,1}_{p,2}})$ that

$$\begin{aligned}
\|F(\sqrt[m]{L})\|_{p\to p} &\le V(X)^{\frac{1}{p}-\frac{1}{2}}\|F(\sqrt[m]{L})P_{B(x_0,1)}\|_{p\to 2}\\
&\le CV(X)^{\frac{1}{p}-\frac{1}{2}}V(X)^{\frac{1}{2}-\frac{1}{p}}N^{n(\frac{1}{p}-\frac{1}{2})}\|\delta_N F\|_{N,q}\\
&\le CN^{n(\frac{1}{p}-\frac{1}{2})}\|\delta_N F\|_{N,q}.
\end{aligned}$$

This means that $(\mathrm{AB_{p,m}})$ is satisfied. This proves Proposition 4.7. □

Remark 4.8. *Suppose that $\mu(X) < \infty$ and $(\mathrm{SC^{q,\kappa}_{p,2,m}})$ holds for some $\kappa \ge 1$. Then $(\mathrm{SC^{\infty}_{p,2,m}})$ and $(\mathrm{G_{p,2,m}})$ are satisfied by Remark 4.5. In addition, $(\mathrm{AB_{p,m}})$ holds by Proposition 4.7. Therefore, Theorem 4.6 holds in this case without assumptions $(\mathrm{G_{p,2,m}})$ and $(\mathrm{AB_{p,m}})$.*

Proof of Theorem 4.6. We consider two cases: $t \ge 1/4$ and $t \le 1/4$.

Case (1). $t \ge 1/4$.

If $t \ge 1/4$ then $\operatorname{supp} \delta_t F \subset [0, 16]$. By $(\mathrm{AB_{p,m}})$,

$$\|F(t\sqrt[m]{L})\|_{p\to p} \le C16^{\kappa n(\frac{1}{p}-\frac{1}{2})+\varepsilon}\|\delta_{16}(F(t\cdot))\|_{16^\kappa,q} \le C\|F\|_\infty.$$

Recall that if $\alpha > 1/q$, then $W^\alpha_q(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $\|F\|_\infty \le C\|F\|_{W^\alpha_q}$. Hence

$$\|F(t\sqrt[m]{L})\|_{p\to p} + \|F(t\sqrt[m]{L})\|_{2\to 2} \le C\|F\|_\infty \le C\|F\|_{W^\alpha_q}.$$

Now (4.22) follows by interpolation.

Case (2). $t \le 1/4$.

Let $\xi \in C_c^\infty$ be an even function such that $\operatorname{supp}\xi \subset [-1/2, 1/2]$, $\hat{\xi}(0) = 1$ and $\hat{\xi}^{(k)}(0) = 0$ for all $1 \le k \le [\alpha] + 2$. Write $\xi_{N^{\kappa-1}} = N^{\kappa-1}\xi(N^{\kappa-1}\cdot)$ where $N = 8[t^{-1}] + 1$ and $[t^{-1}]$ denotes the integer part of t^{-1}. Following [17] we write

$$F(t\sqrt[m]{L}) = (\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F)(\sqrt[m]{L}) + (\xi_{N^{\kappa-1}} * \delta_t F)(\sqrt[m]{L}).$$

Now we prove that

$$\|(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F)(\sqrt[m]{L})\|_{r\to r} \quad \le \quad C\|F\|_{W_q^\alpha}. \tag{4.23}$$

Observe that $\operatorname{supp}(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F) \subseteq [-N, N]$. We apply $(\mathrm{AB_{p,m}})$ to obtain

$$\|(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F)(\sqrt[m]{L})\|_{p\to p} \le CN^{\kappa n(\frac{1}{p}-\frac{1}{2})+\varepsilon}\left\|\delta_N(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F)\right\|_{N^\kappa,q}. \tag{4.24}$$

Everything then boils down to estimate $\|\cdot\|_{N^\kappa,q}$ norm of $\delta_N(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F)$. We make the following claim. For the proof we referee readers to [17, (3.29)] or [22, Propostion 4.6].

Lemma 4.9. *Suppose that* $\xi \in C_c^\infty$ *is an even function such that* $\operatorname{supp}\xi \subset [-1/2, 1/2]$, $\hat{\xi}(0) = 1$ *and* $\hat{\xi}^{(k)}(0) = 0$ *for all* $1 \le k \le [\alpha] + 2$. *Next assume that* $\operatorname{supp} H \subset [-1, 1]$. *Then*

$$\|H - \xi_N * H\|_{N,q} \le CN^{-\alpha}\|H\|_{W_q^\alpha} \tag{4.25}$$

for all $\alpha > 1/q$ *and any* $N \in \mathbb{N}$.

Note that $\delta_N(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F) = \delta_{Nt}F - \xi_{N^\kappa} * \delta_{Nt}F$. Now, if $\alpha > \max\{n(1/p - 1/2), 1/q\}$ then (4.23) follows from Lemma 4.9, estimate (4.24) and the interpolation theorem.

It remains to show that

$$\|(\xi_{N^{\kappa-1}} * \delta_t F)(\sqrt[m]{L})\|_{r\to r} \quad \le \quad C\|F\|_{W_q^\alpha}. \tag{4.26}$$

Let $F^{(\ell)}$ be functions defined in (4.6). we can write

$$(\xi_{N^{\kappa-1}} * \delta_t F)(\lambda) = \sum_{\ell\ge 0}(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)})(\lambda),$$

and hence

$$\left\|(\xi_{N^{\kappa-1}} * \delta_t F)(\sqrt[m]{L})\right\|_{r\to r} \quad \le \quad \sum_{\ell\ge 0}\left\|(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)})(\sqrt[m]{L})\right\|_{r\to r}. \tag{4.27}$$

As in the proof of Theorem 4.2, we fix an $\epsilon > 0$ such that $2n\epsilon(1/p - 1/2) \le \alpha - n(1/p - 1/2)$. For every $t > 0$, and every ℓ, we let $\rho_\ell = 2^{\ell(1+\epsilon)}t$. Let $\psi \in C_c^\infty(1/16, 16)$ such that $\psi(\lambda) = 1$ for $\lambda \in (1/8, 8)$. We decompose

$$\begin{aligned}
(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)})(\sqrt[m]{L})f \quad &= \sum_{i,j:\, d(x_i,x_j)<2\rho_\ell} P_{\widetilde{B}_i}(\delta_t\psi(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)}))(\sqrt[m]{L})P_{\widetilde{B}_j}f \\
&\quad + \sum_{i,j:\, d(x_i,x_j)<2\rho_\ell} P_{\widetilde{B}_i}((1-\delta_t\psi)(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)}))(\sqrt[m]{L})P_{\widetilde{B}_j}f \\
&\quad + \sum_{i,j:\, d(x_i,x_j)\ge 2\rho_\ell} P_{\widetilde{B}_i}(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)})(\sqrt[m]{L})P_{\widetilde{B}_j}f \\
&=: \mathscr{F}_1^{(\ell)}(f) + \mathscr{F}_2^{(\ell)}(f) + \mathscr{F}_3^{(\ell)}(f).
\end{aligned} \tag{4.28}$$

We shall show that

$$\sum_{\ell=0}^{\infty}\left\|\mathscr{F}_i^{(\ell)}(f)\right\|_{r\to r} \le C\|F\|_{W_q^\alpha}, \quad i = 1, 2, 3. \tag{4.29}$$

Similar argument as in (4.12) above give

$$\big\|\mathscr{F}_1^{(\ell)}(f)\big\|_{r\to r} \leq \sup_{x\in X}\{V(x,\rho_\ell)^{\frac{1}{p}-\frac{1}{2}}\big\|(\delta_t\psi(\xi_{N^{\kappa-1}}*\delta_t F^{(\ell)}))(\sqrt[m]{L})P_{B(x,\rho_\ell)}\big\|_{p\to 2}\}\|f\|_r.$$

We then follow the proof of Theorem 3.6 of [11] to get

$$\big\|(\delta_t\psi(\xi_{N^{\kappa-1}}*\delta_t F^{(\ell)}))(\sqrt[m]{L})P_{B(x,\rho_\ell)}\big\|_{p\to 2} \leq CV(x,\rho_\ell)^{\frac{1}{2}-\frac{1}{p}}2^{\ell(1+\epsilon)n(\frac{1}{p}-\frac{1}{2})}\|F^{(\ell)}\|_q.$$

This shows (4.29) for $i=1$ (see (4.15) above).

For $i=2$, the proof of (4.29) is similar to that of (4.17). For $i=3$, we write

$$\big\|\mathscr{F}_3^{(\ell)}(f)\big\|_r^r \leq \sum_i\Big(\sum_{j:\,d(x_i,x_j)\geq 2^{\ell(1+\epsilon)}t}\big\|P_{\widetilde{B}_i}(\xi_{N^{\kappa-1}}*\delta_t F^{(\ell)})(\sqrt[m]{L})P_{\widetilde{B}_j}f\big\|_r\Big)^r.$$

Observe that if $d(x_i,x_j)\geq 2^{\ell(1+\epsilon)}t$, then by Lemma 4.3,

$$\begin{aligned}
&\big\|P_{\widetilde{B}_i}(\xi_{N^{\kappa-1}}*\delta_t F^{(\ell)})(\sqrt[m]{L})P_{\widetilde{B}_j}f\big\|_r\\
&\leq C\|P_{\widetilde{B}_j}f\|_r\int_{-\infty}^{+\infty}|\widehat{\xi_{N^{\kappa-1}}}(\tau)||\phi(2^{-\ell}\tau)\hat{G}(\tau)|\big\|P_{\widetilde{B}_i}e^{(i\tau-1)t^mL}P_{\widetilde{B}_j}\big\|_{r\to r}\,d\tau\\
&\leq Ce^{-c_1 2^{\frac{\epsilon\ell m}{m-1}}}\exp\Big(-c_2\Big(\frac{d(x_i,x_j)}{2^\ell t}\Big)^{\frac{m}{m-1}}\Big)\|F\|_q\|P_{\widetilde{B}_j}f\|_r.
\end{aligned}$$

The rest of the proof of (4.29) for $i=3$ is just a repetition of the proof of (4.19) so we skip it here. This completes the proof of Theorem 4.6. □

5. Hörmander-type spectral multiplier theorems

The aim of this section is to obtain Hörmander-type spectral multiplier theorems to include singular integral versions of Theorems 4.2 and 4.6. We continue with the assumption that (X,d,μ) is a metric measure space satisfying the doubling property and recall that n is the doubling dimension from condition (1.3). Fix a non-trival auxiliary function $\eta\in C_c^\infty(0,\infty)$.

Theorem 5.1. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates* $(\mathrm{DG_m})$ *and condition* $(\mathrm{ST^q_{p,2,m}})$ *for some p,q satisfying $1\leq p<2$ and $1\leq q\leq\infty$. Then for any bounded Borel function F such that $\sup_{t>0}\|\eta\,\delta_t F\|_{W^\alpha_q}<\infty$ for some $\alpha>\max\{n(1/p-1/2),1/q\}$, the operator $F(L)$ is bounded on $L^r(X)$ for all $p<r<p'$. In addition,*

$$\|F(L)\|_{r\to r}\leq C_\alpha\Big(\sup_{t>0}\|\eta\,\delta_t F\|_{W^\alpha_q}+|F(0)|\Big).$$

Proof. Note that by Proposition 2.3, $(\mathrm{ST^q_{p,2,m}})\Rightarrow(\mathrm{ST^\infty_{p,2,m}})\Rightarrow(\mathrm{G_{p,2,m}})$. Now Theorem 5.1 follows from Theorems 3.3 and 4.2. □

Note that Gaussian bounds $(\mathrm{GE_m})$ implies estimates $(\mathrm{DG_m})$ and $(\mathrm{G_{p,2,m}})$ so by Proposition 2.3, $(\mathrm{ST^\infty_{p,2,m}})$ holds for $q=\infty$. This means that one can omit conditions $(\mathrm{DG_m})$ and $(\mathrm{ST^q_{p,2,m}})$ in Theorem 5.1 if the case $q=\infty$ is considered. We describe the details in Theorem 5.2 below.

Theorem 5.2. *Let L be a non-negative self-adjoint operator on $L^2(X)$ satisfying Gaussian estimates* $(\mathrm{GE_m})$*. Let $1\leq p<2$.*

Then for any bounded Borel function F such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W^\alpha_\infty} < \infty$ *for some* $\alpha > n(1/p - 1/2)$ *the operator* $F(L)$ *is bounded on* $L^r(X)$ *for all* $p < r < p'$. *In addition,*

$$\|F(L)\|_{r\to r} \le C_\alpha\Big(\sup_{t>0} \|\eta\, \delta_t F\|_{W^\alpha_\infty} + |F(0)|\Big).$$

Proof. Theorem 5.2 follows Proposition 2.3 and Theorem 5.1. ◻

The next theorem is a variation of Theorem 5.1 suitable for the operators satisfying condition $(\mathrm{SC}^{q,\kappa}_{p,2,m})$.

Theorem 5.3. *Suppose the operator L satisfies Davies-Gaffney estimates* $(\mathrm{DG_m})$, *conditions* $(\mathrm{G_{p,2,m}})$ *and* $(\mathrm{SC}^{q,\kappa}_{p,2,m})$ *for some p, q such that $1 \le p < 2$ and $1 \le q \le \infty$, and a fixed natural number κ. In addition, we assume that for any $\varepsilon > 0$ there exists a constant C_ε such that for all $N \in \mathbb{N}$ and all even Borel functions F such that* $\mathrm{supp}\, F \subset [-N, N]$,

$$\|F(\sqrt[m]{L})\|_{p\to p} \le C_\varepsilon N^{\kappa n(\frac{1}{p}-\frac{1}{2})+\varepsilon} \|\delta_N F\|_{N^\kappa, q}. \tag{$\mathrm{AB_{p,m}}$}$$

Then for any even bounded Borel function F such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W^\alpha_q} < \infty$ *for some* $\alpha > \max\{n(1/p - 1/2), 1/q\}$ *the operator $F(L)$ is bounded on $L^r(X)$ for all $p < r < p'$. In addition,*

$$\|F(L)\|_{r\to r} \le C_\alpha\Big(\sup_{t>0} \|\eta\, \delta_t F\|_{W^\alpha_q} + |F(0)|\Big).$$

Proof. Theorem 5.3 follows Theorems 3.3 and 4.6. ◻

Remark 5.4. *Suppose that $\mu(X) < \infty$ and* $(\mathrm{SC}^{q,\kappa}_{p,2,m})$ *holds for some $\kappa \ge 1$. Then* $(\mathrm{SC}^{\infty}_{p,2,m})$ *and* $(\mathrm{G_{p,2,m}})$ *are satisfied by Remark 4.5. In addition,* $(\mathrm{AB_{p,m}})$ *holds by Proposition 4.7. Therefore, Theorem 5.3 holds in this case without assumptions* $(\mathrm{G_{p,2,m}})$ *and* $(\mathrm{AB_{p,m}})$.

6. Applications

As an illustration of our results we shall discuss a few of possible applications. Our main results, Theorems 4.2, 4.6, 5.1 and 5.3, can be applied to all examples which are discussed in [22] and [11]. Those include the standard Laplace operator; Laplace-Beltrami operator acting on compact manifolds; the Laplace-Beltrami operator and some of its perturbation on asymptotically conic manifolds, see [28]; the harmonic oscillator and its perturbations; homogeneous sub-Laplacians on nilpotent Lie groups. We do not discuss the details here as the obtained corollaries coincide with applications described in [11], except that we are not able to prove endpoints estimates for Bochner-Riesz sumability. We suspect that endpoints results possibly do not hold in m-th order operators setting.

6.1. m-th order differential operators on compact manifolds.

For a general positive definite elliptic operator on a compact manifold, condition $(\mathrm{GE_m})$ holds by general elliptic regularity theory. As a consequence of Theorem 5.3, we obtain alternative proof of Theorem 3.2 in [42] described by A. Seeger and C. Sogge. The result can be stated in the following way.

Theorem 6.1. *Let M be a compact connected manifold without boundary of dimension $n \ge 2$. Let $P_m(x, D)$ be a positive definite elliptic pseudo-differential operator of order m on M. Suppose that for each $x \in M$, the cosphere*

$$\Sigma_x = \{\xi \in T^*_x M \backslash 0 : P_m(x, \xi) = 1\} \tag{6.1}$$

has nonzero Gaussian curvature everywhere, where $P_m(x, \xi)$ is the principal symbol. Let $1 \le p \le 2(n + 1)/(n + 3)$ and $1 \le q \le \infty$. Then for any even bounded Borel function F such that

$\sup_{t>0}\|\eta\,\delta_t F\|_{W^\alpha_q} < \infty$ *for some* $\alpha > \max\{n(1/p-1/2), 1/q\}$*, the operator* $F(P_m)$ *is bounded on* $L^r(X)$ *for all* $p < r < p'$*. In addition,*

$$\|F(P_m)\|_{r\to r} \le C_\alpha \sup_{t>0}\|\eta\,\delta_t F\|_{W^\alpha_q}.$$

Proof. Under the non-degenerate assumption of the cospheres Σ_x, it follows by Corollary 2.2 of [43] that estimates ($\mathrm{SC}^{2,1}_{\mathrm{p},2}$) hold for $1 \le p \le 2(n+1)/(n+3)$. Then the result is a consequence of Theorem 5.3 and Remark 5.4. □

6.2. ***m*-th order elliptic differential operators with constant coefficients.** Let $P_m(D)$ be a real homogeneous elliptic polynomial of order m on $\mathbb{R}^n$, $n \ge 2$, and Σ is a hypersurface defined by

$$\Sigma = \{\xi \in \mathbb{R}^n : |P_m(\xi)| = 1\}, \tag{6.2}$$

where $P_m(\xi)$ is the symbol. Recall that Σ is of finite type if there exist $k \in \mathbb{N}$ and $C > 0$ such that

$$\sum_{j=1}^{k} \left|\langle \eta, \nabla\rangle^j P_m(\xi)\right| \ge C > 0, \quad \xi \in \Sigma \ \text{ and } \ \eta \in \mathbf{S}^{n-1}, \tag{6.3}$$

where $\langle \eta, \nabla\rangle = \sum_{i=1}^n \eta_i \partial/\partial x_i$. The least k in (6.3) is called the type order of Σ. Say that Σ is convex if

$$\Sigma \subseteq \{\eta \in \mathbb{R}^n \big|\ \langle \eta - \xi, \nabla P_m(\xi)\rangle \ge 0\}, \quad \xi \in \Sigma \tag{6.4}$$

or

$$\Sigma \subseteq \{\eta \in \mathbb{R}^n \big|\ \langle \eta - \xi, \nabla P_m(\xi)\rangle \le 0\}, \quad \xi \in \Sigma. \tag{6.5}$$

For a given P_m, we know that the corresponding Σ is always of finite type and $2 \le k \le m$. But it is obviously not always convex. The hypersurface Σ is convex and $k = 2$ if and only if Σ has nonzero Gaussian curvature everywhere. A simple example of polynomials whose level hypersurface Σ is of type m is $\xi_1^m + \cdots + \xi_n^m (m = 4, 6, \cdots)$. We notice that there exist polynomials P_m whose level hypersurfaces Σ are of type $k(< m)$. For example, when $P_6(\xi) = \xi_1^6 + 5\xi_1^2\xi_2^4 + \xi_2^6$, the corresponding hypersurface Σ is of type 4, but $m = 6$ (see [20]).

Proposition 6.2. *Let* $P_m(D)$ *be a real homogeneous elliptic polynomial of order* m *on* $\mathbb{R}^n, n \ge 2$*. Suppose* Σ *is a convex hypersurface of finite type* k *for* $2 \le k \le m$ *and that* $1 \le p \le 2(n-1+k)/(n-1+2k)$*. Alternatively assume that* Σ *has nonzero Gaussian curvature everywhere and that* $1 \le p \le 2(n+1)/(n+3)$*. Then we have*

$$\|dE_{\sqrt[m]{L}}(\lambda)\|_{p\to p'} \le C\lambda^{n(\frac{1}{p}-\frac{1}{p'})-1}, \quad \lambda > 0. \tag{6.6}$$

Hence condition ($\mathrm{ST}^2_{\mathrm{p},2,\mathrm{m}}$) *holds.*

Proof. Estimates (6.6) and ($\mathrm{ST}^2_{\mathrm{p},2,\mathrm{m}}$) follow from Theorem B in [9] and Theorem 1 in [27]. □

We are now able to state the following result describing spectral multipliers for m-th order elliptic differential operators with constant coefficients.

Theorem 6.3. *Suppose Σ is a convex hypersurface of finite type k for $2 \le k \le m$ and that $1 \le p \le 2(n-1+k)/(n-1+2k)$. Alternatively assume that Σ has nonzero Gaussian curvature everywhere and that $1 \le p \le 2(n+1)/(n+3)$. Then for any even bounded Borel function F such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W_q^\alpha} < \infty$ for some $\alpha > n(1/p - 1/2)$ and $1 \le q \le \infty$, the operator $F(P_m)$ is bounded on $L^r(X)$ for all $p < r < p'$. In addition,*

$$\|F(P_m)\|_{r\to r} \le C_\alpha \sup_{t>0} \|\eta\, \delta_t F\|_{W_q^\alpha}.$$

Proof. It is known that the semigroup e^{-tL} has integral kernels $p_t(x,y)$ satisfying the following estimates ($\mathrm{GE_m}$) (see [19]). Now Theorem 6.3 is a straightforward consequence of Proposition 6.2 and Theorem 5.1. □

6.3. **Biharmonic operators with rough potentials.** In the section we consider the biharmonic operator $\Delta^2 = (\partial_1^2 + \partial_2^2 + \partial_3^2)^2$ acting on $L^2(\mathbb{R}^3)$. Assume now that $n = 3$ and V is a real-valued measurable function such that $0 \le V \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. We define a self-adjoint operator L as Friedrich's extension of the operator $\Delta^2 + V$ initially defined on $C_c^\infty(\mathbb{R}^3)$.

To be able to apply our results to the operator L we first show that the corresponding semigroup satisfies Davies-Gaffney estimates ($\mathrm{DG_4}$) and 4-th order Gaussian bounds ($\mathrm{GE_4}$).

Proposition 6.4. *Suppose that $0 \le V \in L^1(\mathbb{R}^3)$. Then the semigroup generated by the operator $H = \Delta^2 + V$ and the corresponding heat kernel $p_t(x,y)$ satisfies Davies-Gaffney estimates* ($\mathrm{DG_4}$) *and Gaussian bounds* ($\mathrm{GE_4}$).

Proof. Following [23] we consider the set of linear functions $\psi\colon \mathbb{R}^3 \to \mathbb{R}$ of the form $\psi(x) = a \cdot x$, where $a = (a_1, a_2, a_3) \in \mathbf{S}^2$. Then for $\lambda \in \mathbb{R}$ we consider perturbed operator

$$H_{\lambda\psi} = e^{-\lambda\psi} H e^{\lambda\psi} = e^{-\lambda\psi} \Delta^2 e^{\lambda\psi} + V = \Delta^2_{\lambda\psi} + V,$$

where $\Delta_{\lambda\psi} = e^{-\lambda\psi} \Delta e^{\lambda\psi} = (\partial_1 + a_1\lambda)^2 + (\partial_2 + a_2\lambda)^2 + (\partial_3 + a_3\lambda)^2$, see [23, Lemma 10]. Note that

$$\exp(-tH_{\lambda\psi}) = e^{-\lambda\psi} \exp(-tH) e^{\lambda\psi}.$$

By Lemma 10 of [23] there exists a constant $c > 0$ such that

$$\|\exp(-t\Delta^2_{\lambda\psi})\|_{2\to 2} = e^{c\lambda^4 t}.$$

However we assume that $V \ge 0$ so

$$\|e^{-\lambda\psi} \exp(-tH) e^{\lambda\psi}\|_{2\to 2} \le e^{c\lambda^4 t},$$

see also [19]. Now consider $a = (a_1, a_2, a_3) \in \mathbf{S}^2$ such that $\psi(x) - \psi(y) = |x - y|$. Then

$$\left\| P_{B(x,t^{1/4})} e^{-tH} P_{B(y,t^{1/4})} \right\|_{2\to 2} \le e^{c\lambda^4 t - \lambda(|x-y| - 2t^{1/4})}.$$

Taking infimum over λ in the above inequality proves estimates ($\mathrm{DG_4}$).

To prove Gaussian estimates ($\mathrm{GE_4}$) we first note that $\|(I + t\Delta^2)^{-1/2}\|_{2\to\infty} \le Ct^{-3/4}$. However we assume that $V(x) \ge 0$ for all $x \in \mathbb{R}^3$ so

$$\langle (I + t(\Delta^2 + V)) f, f\rangle \ge \langle (I + t\Delta^2) f, f\rangle = \|(I + t\Delta^2)^{-1/2} f\|_2.$$

Hence $\|(I + tH)^{-1/2}\|_{2\to\infty} \le Ct^{-3/4}$ and

$$\|\exp(-tH)\|_{2\to\infty} \le \|(I + tH)^{-1/2}\|_{2\to\infty} \|(I + tH)^{-1/2} \exp(-tH)\|_{2\to 2} \le Ct^{-3/4}.$$

This proves on-diagonal bounds for the corresponding heat kernel. Now it suffices to ensure that all assumptions of our abstract results hold. To prove off-diagonal Gaussian bounds we note that by formula (9) of [23] for some constant $c > 0$

$$\mathrm{Re}\langle H_{\lambda\psi} f, f\rangle \geq \langle(\Delta^2 - \lambda^4 c) f, f\rangle.$$

Now standard heat kernels theory argument shows

$$\|e^{-\lambda\psi}\exp(-tH)e^{\lambda\psi}\|_{2\to\infty} \leq Ct^{-3/4}e^{c\lambda^4 t}.$$

This estimate implies off-diagonal Gaussian bounds (GE_4) by Davies' perturbation argument. □

Next we establish restriction type estimates for spectral measure $dE_{\sqrt[4]{H}}(\lambda)$ associated with the special higher order operator $H = \Delta^2 + V$ with potentials V on $\mathbb{R}^3$. Let $H_0 = \Delta^2$ be the self-adjoint extension operator on $L^2(\mathbb{R}^3)$. Then we have $\sigma(H_0) = [0, \infty)$. For any $z \in \mathbb{C}\backslash\sigma(H_0)$, the resolvent

$$R_0(z) = (H_0 - z)^{-1}$$

is well-defined on $L^2(\mathbb{R}^3)$. We consider the boundary behavior of $R_0(z)$ as z approaches to some $\lambda > 0$ since it is connected with the spectral measure by the limiting absorption principle:

$$\frac{1}{2\pi i}\langle(R_0(\lambda + i0) - R_0(\lambda - i0))f,\ g\rangle = \langle dE_{H_0}(\lambda)f,\ g\rangle,\ \ f, g \in \mathscr{S}(\mathbb{R}^3). \tag{6.7}$$

Let $\mu = \lambda + i\varepsilon$ where $\lambda > 0$ and $0 < \varepsilon < \frac{\lambda}{10}$. By elementary integration, it can be verfied that

$$\int_{\mathbb{R}^3} \frac{e^{-i\xi x}}{|\xi|^2 - \mu^2}\, d\xi = \frac{e^{i\mu|x|}}{4\pi|x|} \quad \text{and} \quad \int_{\mathbb{R}^3} \frac{e^{-i\xi x}}{|\xi|^2 + \mu^2}\, d\xi = \frac{e^{-\mu|x|}}{4\pi|x|},$$

which gives

$$\int_{\mathbb{R}^3} \frac{e^{-i\xi x}}{|\xi|^4 - \mu^4}\, d\xi = \frac{1}{2\mu^2}\Big(\frac{e^{i\mu|x|}}{4\pi|x|} - \frac{e^{-\mu|x|}}{4\pi|x|}\Big) = \frac{e^{i\mu|x|}}{4(1+i)\pi\mu}\Big(\frac{1 - e^{-(1-i)\mu|x|}}{(1-i)\mu|x|}\Big). \tag{6.8}$$

Hence $K(\mu^4, x)$ - the Green kernel of $(\Delta^2 - \mu^4)^{-1}$ is given by the formula

$$K(\mu^4, x) = \frac{e^{i\mu|x|}}{4(1+i)\pi\mu}\Big(\frac{1 - e^{-(1-i)\mu|x|}}{(1-i)\mu|x|}\Big). \tag{6.9}$$

Proposition 6.5. *Let $R_0(\mu^4) = (H_0 - \mu^4)^{-1}$ where $H_0 = \Delta^2$ acts on $\mathbb{R}^3$. If $\mu = \lambda + i\varepsilon$ with $\lambda > 0$ and $0 < |\varepsilon| < \frac{\lambda}{10}$, then for every $1 \leq p \leq \frac{4}{3}$,*

$$\|R_0(\mu^4)\|_{p\to p'} \leq C|\mu|^{3(\frac{1}{p} - \frac{1}{p'}) - 4}. \tag{6.10}$$

In particular, the following estimates of incoming and outcoming operators are satisfied

$$\|R_0(\lambda^4 - i0)\|_{p\to p'} = \|R_0(\lambda^4 + i0)\|_{p\to p'} \leq C\lambda^{3(\frac{1}{p} - \frac{1}{p'}) - 4}. \tag{6.11}$$

Proof. Observe that by (6.8) and (6.9), there exists a constant $C > 0$ such that $|K(\mu^4, x)| \leq C|\mu|^{-1}$. By Young's inequality,

$$\|R_0(\mu^4)f\|_\infty \leq C|\mu|^{-1}\|f\|_1.$$

Now by the interpolation (see [3]), it suffices to verify (6.10) for $p = 4/3$, that is, $\|R_0(\mu^4)f\|_4 \leq C|\mu|^{-5/2}\|f\|_{4/3}$. Observe that $|\mu| \sim \lambda$, by the scaling in λ it reduces to show that uniformly

$$\|R_0((1 + i\epsilon)^4)f\|_4 \leq C\|f\|_{4/3},\ \ 0 < \epsilon < 1/10. \tag{6.12}$$

Now one can write

$$R_0((1+i\epsilon)^4) = \frac{1}{2(1+i\epsilon)^2}((-\Delta-(1+i\epsilon)^2)^{-1} - (-\Delta+(1+i\epsilon)^2)^{-1}).$$

To estimate $L^{4/3}$ to L^4 norm of $(-\Delta-(1+i\epsilon)^2)^{-1}$, one can use the argument from the proof of Theorem 2.3 in [33], see also Lemma 4 in [26]. $L^{4/3}$ to L^4 norm estimates of $(-\Delta+(1+i\epsilon)^2)^{-1}$ are straightforward consequence of the standard Gaussian bounds. □

Proposition 6.6. *Let $R_0(\mu^4) = (H_0-\mu^4)^{-1}$ where $H_0 = \Delta^2$ acts on $\mathbb{R}^3$ and $\mu = \lambda + i\varepsilon$ where $\lambda > 0$ and $0 \le |\varepsilon| < \frac{\lambda}{10}$. Suppose that $V \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$.*

Then

(i) *For $4 \le p \le \infty$, the operator map $\mu \mapsto R_0(\mu^4)V$ is continuous from the cone domain $\{\mu = \lambda \pm i\varepsilon, \lambda > 0$ and $0 \le \varepsilon < \frac{\lambda}{10}\}$ to the space of bounded operators on $L^p(\mathbb{R}^3)$.*

(ii) *For $4 \le p \le \infty$ there exists a positive constant λ_0 such that for all $\lambda \ge \lambda_0 > 0$ the operator $I + R_0(\mu^4)V$ is invertible on $L^p(\mathbb{R}^3)$ and*

$$\sup_{\lambda\ge\lambda_0} \|(I+R_0(\mu^4)V)^{-1}\|_{p\to p} \le C.$$

Proof. Note that, for the case, we can write the $R_0(\mu^4)V$ into the following two parts:

$$R_0(\mu^4)V = ((-\Delta-\mu^2)^{-1}V - (-\Delta+\mu^2)^{-1}V)/2\mu^2.$$

Hence the proof of (i) follows from Lemmas 8 and 10 of [26].

Next we prove (ii). Define the operator M_V by the formula $M_V f(x) = V(x)f(x)$ and we note that if $V \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ then $\|M_V\|_{p\to p'} < \infty$ for all $4 \le p \le \infty$. Now by Proposition 6.5 there exists a constant $\lambda_0 > 0$

$$\|R_0(\mu^4)V\|_{p\to p} \le \|R_0(\mu^4)\|_{p'\to p}\|M_V\|_{p\to p'} \le \frac{1}{2}$$

for all $\lambda \ge \lambda_0$. This proves (ii) and concludes the proof of Proposition 6.6. □

Proposition 6.7. *Suppose that $H = \Delta^2 + V$ on $\mathbb{R}^3$ with a real-valued $V \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$.*

Then there exists a $\lambda_0 > 0$ such that

$$\|dE_{\sqrt[4]{H}}(\lambda)\|_{L^p\to L^{p'}} \le C\lambda^{3(\frac{1}{p}-\frac{1}{p'})-1}$$

for all $\lambda \ge \lambda_0$ and $1 \le p \le 4/3$.

Proof. Let $\mu = \lambda + i\varepsilon$ where $\lambda \ge \lambda_0 > 0$ and $0 < |\varepsilon| < \frac{\lambda}{10}$. We denote by $R(\mu^4) = (H-\mu^4)^{-1}$ the resolvent of $H = \Delta^2 + V$ on $L^2(\mathbb{R}^3)$. Note that

$$R(\mu^4) = (I+R_0(\mu^4)V)^{-1}R_0(\mu^4). \tag{6.13}$$

Hence it follows that from Propositions 6.5 and 6.6

$$\|R(\mu^4)f\|_{p'} \le \|(I+R_0(\mu^4)V)^{-1}\|_{p'\to p'}\|R_0(\mu^4)f\|_{p'} \le C(\lambda_0, V)\,|\mu|^{3(\frac{1}{p}-\frac{1}{p'})-1}\,\|f\|_p.$$

By the limit absorption principle the above estimates imply Proposition 6.7. □

Finally, we are now able to state the following results describing spectral multipliers for the biharmonic operators with some potential V on $\mathbb{R}^3$

Theorem 6.8. *Suppose that $H = \Delta^2 + V$ on $\mathbb{R}^3$ with a positive real-valued $0 \le V \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and that $1 \le p \le 4/3$. Next assume that F is a bounded Borel function such that $\operatorname{supp} F \subseteq [1/4, 4]$ and $F \in W_2^\alpha(\mathbb{R})$ for some $\alpha > 3(1/p - 1/2)$. Then for every $p < r \le p'$, $F(tH)$ is bounded on $L^r(X)$ for all $t > 0$. In addition*

$$\sup_{t<1/(16\lambda_0)} \|F(tH)\|_{r\to r} \le C_r \|F\|_{W_2^\alpha}, \tag{6.14}$$

and

$$\sup_{t\ge 1/(16\lambda_0)} \|F(tH)\|_{r\to r} \le C_r \|F\|_{W_\infty^\alpha} \tag{6.15}$$

for some constant $\lambda_0 > 0$ as in Proposition 6.7.

Proof. The result is a straightforward consequence of Propositions 6.4, 6.7, 4.1 and Theorem 4.2. □

6.4. **Laplace type operators acting on fractals.** Theorem 5.2 can be applied to any operator which satisfies estimates ($\mathrm{GE_m}$) and for which the ambient spaces satisfies the doubling condition. A compelling class of such operators is considered in the theory of diffusion processes on fractals, see for example [2, 8, 34, 47, 48]. One of the most well known space of this type is Sierpiński gasket SG see for example [34, 48]. The Laplace operator on the Sierpiński gasket SG (Neumann or Dirichlet) satisfies Gaussian bound of order $m = \log 5/(\log 5 - \log 3)$ and (2.2) holds with with the homogeneous dimension given by $n = \log 3/(\log 5 - \log 3) = 2.1506601\ldots$, see [2, 47, 48]. Now application of Theorem 5.2 to this setting yields the following result.

Theorem 6.9. *Suppose that L is the Laplacian on the Sierpiński gasket. Let $1 \le p < 2$. Then for any bounded Borel function F such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W_\infty^\alpha} < \infty$ for some $\alpha > n(1/p - 1/2)$, the operator $F(L)$ is bounded on $L^r(X)$ for all $p < r < p'$. In addition,*

$$\|F(L)\|_{r\to r} \le C_\alpha \Big(\sup_{t>0} \|\eta\, \delta_t F\|_{W_\infty^\alpha} + |F(0)| \Big),$$

where $n = \log 3/(\log 5 - \log 3)$ and $m = \log 5/(\log 5 - \log 3)$.

Proof. The result is direct consequence of Theorem 5.2. □

We do not know however if Theorem 6.9 is sharp. The case $p = 1$ of this result is discussed in details in [22]. Theorem 6.9 can be extend to include broader class of fractals. One simple class of possible generalization can be given by products of any number of copies of Sierpiński gaskets.

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